

# On the global existence and blowup of smooth solutions of 3-D compressible Euler equations with time-dependent damping

Fei Hou<sup>1,\*</sup>, Ingo Witt<sup>2,\*</sup>, Huicheng Yin<sup>3,\*</sup>

1. Department of Mathematics and IMS, Nanjing University, Nanjing 210093, China

2. Mathematical Institute, University of Göttingen, Bunsenstr. 3-5, 37073 Göttingen, Germany

3. School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China

October 16, 2015

## Abstract

In this paper, we are concerned with the global existence and blowup of smooth solutions of the 3-D compressible Euler equation with time-dependent damping

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + p \mathbf{I}_3) = -\frac{\mu}{(1+t)^\lambda} \rho u, \\ \rho(0, x) = \bar{\rho} + \varepsilon \rho_0(x), \quad u(0, x) = \varepsilon u_0(x), \end{cases}$$

where  $x \in \mathbb{R}^3$ ,  $\mu > 0$ ,  $\lambda \geq 0$ , and  $\bar{\rho} > 0$  are constants,  $\rho_0, u_0 \in C_0^\infty(\mathbb{R}^3)$ ,  $(\rho_0, u_0) \not\equiv 0$ ,  $\rho(0, \cdot) > 0$ , and  $\varepsilon > 0$  is sufficiently small. For  $0 \leq \lambda \leq 1$ , we show that there exists a global smooth solution  $(\rho, u)$  when  $\operatorname{curl} u_0 \equiv 0$ , while for  $\lambda > 1$ , in general, the solution  $(\rho, u)$  will blow up in finite time. Therefore,  $\lambda = 1$  appears to be the critical value for the global existence of small amplitude smooth solutions.

**Keywords.** Compressible Euler equations, damping, time-weighted energy inequality, Klainerman-Sobolev inequality, blowup.

**2010 Mathematical Subject Classification.** 35L70, 35L65, 35L67, 76N15.

## 1 Introduction

In this paper, we are concerned with the global existence and blowup of smooth solutions of the three-dimensional compressible Euler equations with time-dependent damping

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + p \mathbf{I}_3) = -\frac{\mu}{(1+t)^\lambda} \rho u, \\ \rho(0, x) = \bar{\rho} + \varepsilon \rho_0(x), \quad u(0, x) = \varepsilon u_0(x), \end{cases} \quad (1.1)$$

---

\*Fei Hou (houfeimath@gmail.com) and Huicheng Yin (huicheng@nju.edu.cn) were supported by the NSFC (No. 11571177) and the Priority Academic Program Development of Jiangsu Higher Education Institutions. Ingo Witt (iwitt@uni-math.gwdg.de) was partly supported by the DFG via the Sino-German project ‘‘Analysis of Partial Differential Equations and Applications.’’ This research was carried out when Huicheng Yin and Fei Hou were visiting the Mathematical Institute of the University of Göttingen in February 2013 and November 2014, respectively.

where  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ ,  $\rho, u = (u_1, u_2, u_3)$ , and  $p$  stand for the density, velocity, and pressure, respectively,  $I_3$  is the  $3 \times 3$  identity matrix, and  $u_0 = (u_{1,0}, u_{2,0}, u_{3,0})$ . The equation of state of the gas is assumed to be  $p(\rho) = A\rho^\gamma$ , where  $A > 0$  and  $\gamma > 1$  are constants. Furthermore,  $\mu > 0$ ,  $\lambda \geq 0$ , and  $\bar{\rho} > 0$  are constants,  $\rho_0, u_0 \in C_0^\infty(\mathbb{R}^3)$ ,  $(\rho_0, u_0) \not\equiv 0$ ,  $\rho(0, \cdot) > 0$ , and  $\varepsilon > 0$  is sufficiently small.

For  $\mu = 0$ , (1.1) is the standard compressible Euler equation. It is well known that smooth solutions  $(\rho, u)$  of (1.1) will in general blow up in finite time. For the extensive literature on blowup results and the blowup mechanism for  $(\rho, u)$ , see [1–6, 17, 18, 20, 25, 26] and the references therein.

For  $\lambda = 0$ , it has been shown that (1.1) admits a global smooth solution and the long-term behavior of the solution  $(\rho, u)$  has been established, see [12, 16, 19, 21, 22].

For  $\mu > 0$  and  $\lambda > 0$ , an interesting problem arises: does the smooth solution of (1.1) blow up in finite time or does it exist globally? In this paper, we will systematically study this problem under the assumption that  $\text{curl } u_0 = (\partial_2 u_{3,0} - \partial_3 u_{2,0}, \partial_3 u_{1,0} - \partial_1 u_{3,0}, \partial_1 u_{2,0} - \partial_2 u_{1,0}) \equiv 0$ . In this case it is not hard to see that  $\text{curl } u(t, \cdot) \equiv 0$  for all  $t \geq 0$  as long as the smooth solution  $(\rho, u)$  of (1.1) exists. Then one can introduce a potential function  $\varphi = \varphi(t, x)$  such that  $u = \nabla \varphi$  (here and below,  $\nabla = \nabla_x$ ), where the  $C^\infty$  scalar function  $\varphi$  has compact support in  $x$  (as  $u(t, \cdot)$  has compact support in view of  $u_0 \in C_0^\infty(\mathbb{R}^3)$  and finite propagation speed which holds for hyperbolic systems). Substituting  $u = \nabla \varphi$  into the second equation of (1.1), we obtain

$$\partial_t \varphi + \frac{1}{2} |\nabla \varphi|^2 + h(\rho) + \frac{\mu}{(1+t)^\lambda} \varphi = 0, \quad (1.2)$$

where  $h'(\rho) = c^2(\rho)/\rho$  with  $c(\rho) = \sqrt{p'(\rho)}$  and  $h(\bar{\rho}) = 0$ . From  $h'(\rho) > 0$  for  $\rho > 0$  we have that

$$\rho = h^{-1} \left( - \left( \partial_t \varphi + \frac{1}{2} |\nabla \varphi|^2 + \frac{\mu}{(1+t)^\lambda} \varphi \right) \right), \quad (1.3)$$

where  $\bar{\rho} = h^{-1}(0)$  and  $h^{-1}$  is the inverse function of  $h = h(\rho)$ . Substituting (1.3) into the first equation of (1.1) yields

$$\begin{aligned} \partial_t^2 \varphi - c^2(\rho) \Delta \varphi + 2 \sum_{k=1}^3 (\partial_k \varphi) \partial_{tk} \varphi + \sum_{i,k=1}^3 (\partial_i \varphi) (\partial_k \varphi) \partial_{ik} \varphi \\ + \frac{\mu}{(1+t)^\lambda} |\nabla \varphi|^2 + \partial_t \left( \frac{\mu}{(1+t)^\lambda} \varphi \right) = 0. \end{aligned} \quad (1.4)$$

As for the initial data  $\varphi(0, \cdot)$  and  $\partial_t \varphi(0, \cdot)$  for Eq. (1.4): Obviously,  $\varphi(0, \cdot) = \varepsilon \varphi_0$ , where  $\varphi_0(x) = \int_{-\infty}^{x_1} u_{1,0}(s, x_2, x_3) ds$ . Note that  $\varphi_0 \in C_0^\infty(\mathbb{R}^3)$  in view of  $\text{curl } u_0 \equiv 0$  and  $u_0 \in C_0^\infty(\mathbb{R}^3)$ . Furthermore, from Eq. (1.2) we infer that  $\partial_t \varphi(0, \cdot) = \varepsilon \varphi_1 + \varepsilon^2 g(\cdot, \varepsilon)$ , where  $\varphi_1 = - \left( \mu \varphi_0 + \frac{c^2(\bar{\rho})}{\bar{\rho}} \rho_0 \right)$  and

$$g(x, \varepsilon) = -\rho_0^2(x) \int_0^1 \left( \frac{c^2(\rho)}{\rho} \right)' \Big|_{\rho=\bar{\rho}+\theta\varepsilon\rho_0(x)} d\theta - \frac{1}{2} \sum_{i=1}^3 u_{i,0}^2(x).$$

Notice that  $g(x, \varepsilon)$  is smooth in  $(x, \varepsilon)$  and has compact support in  $x$ . Consequently, studying prob-

lem (1.1) under the assumption  $\text{curl } u_0 \equiv 0$  is equivalent to investigating the problem

$$\begin{cases} \partial_t^2 \varphi - c^2(\rho) \Delta \varphi + 2 \sum_{k=1}^3 (\partial_k \varphi) \partial_{tk} \varphi + \sum_{i,k=1}^3 (\partial_i \varphi) (\partial_k \varphi) \partial_{ik} \varphi \\ \quad + \frac{\mu}{(1+t)^\lambda} |\nabla \varphi|^2 + \partial_t \left( \frac{\mu}{(1+t)^\lambda} \varphi \right) = 0, \\ \varphi(0, x) = \varepsilon \varphi_0(x), \quad \partial_t \varphi(0, x) = \varepsilon \varphi_1(x) + \varepsilon^2 g(x, \varepsilon). \end{cases} \quad (1.5)$$

Here we mention that

$$c^2(\rho) = c^2(\bar{\rho}) - (\gamma - 1) \left( \partial_t \varphi + \frac{1}{2} |\nabla \varphi|^2 + \frac{\mu}{(1+t)^\lambda} \varphi \right)$$

which follows by direct computation.

We now state the first main result of this paper.

**Theorem 1.1** (Global existence for  $0 \leq \lambda \leq 1$ ). *Suppose that  $\text{curl } u_0 \equiv 0$ . If  $\mu > 0$  and  $0 \leq \lambda \leq 1$ , then, for  $\varepsilon > 0$  small enough, (1.5) admits a global smooth solution  $\varphi$ . As a consequence, (1.1) has a global smooth solution  $(\rho, u)$  which fulfills  $\rho > 0$  and which is uniformly bounded for  $t \geq 0$  together with all its derivatives.*

*Remark 1.1.* The principal part of the linearization of the equation in (1.5) about  $(\rho, \varphi) = (\bar{\rho}, 0)$  is

$$\mathcal{L}(\dot{\varphi}) \equiv \partial_t^2 \dot{\varphi} - c^2(\bar{\rho}) \Delta \dot{\varphi} + \frac{\mu}{(1+t)^\lambda} \partial_t \dot{\varphi} - \frac{\mu \lambda}{(1+t)^{\lambda+1}} \dot{\varphi}. \quad (1.6)$$

For the linear operator  $\mathcal{L}_0$  with

$$\mathcal{L}_0(\dot{\varphi}) \equiv \partial_t^2 \dot{\varphi} - c^2(\bar{\rho}) \Delta \dot{\varphi} + \frac{\mu}{(1+t)^\lambda} \partial_t \dot{\varphi},$$

which appears as part of (1.6), it is shown in [23, 24] that the large-term behavior of solutions  $\varphi$  of  $\mathcal{L}_0(\dot{\varphi}) = 0$  depends on the value of  $\lambda$ . For  $0 \leq \lambda < 1$  is the same as the large-term behavior of solutions of the linear heat equation  $\partial_t \dot{\varphi} - c^2(\bar{\rho}) \Delta \dot{\varphi} = 0$ , while for  $\lambda > 1$  it is the same as the large-term behavior of solutions of the linear wave equation  $\partial_t^2 \dot{\varphi} - c^2(\bar{\rho}) \Delta \dot{\varphi} = 0$ . Moreover, precise microlocal large-term decay properties of solutions  $\dot{\varphi}$  of  $\mathcal{L}(\dot{\varphi}) = 0$  have been established in [10] for a special range of values of  $\lambda$  and  $\mu$ . It seems to be difficult, however, to apply these microlocal estimates to attack the quasilinear problem (1.5). (In general, those microlocal estimates are useful when treating semilinear damped wave equations, see [8, 9] and references therein.)

*Remark 1.2.* For the 1-D Burgers equation with damping term

$$\begin{cases} \partial_t w + w \partial_x w = - \frac{\mu}{(1+t)^\lambda} w, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ w(0, x) = \varepsilon w_0(x), \end{cases} \quad (1.7)$$

where  $\mu > 0$  and  $\lambda \geq 0$  are constants,  $w_0 \in C_0^\infty(\mathbb{R})$ ,  $w_0 \not\equiv 0$ , and  $\varepsilon > 0$  is sufficiently small, one concludes by the method of characteristics that

$$\begin{cases} T_\varepsilon = \infty & \text{if } 0 \leq \lambda < 1 \text{ or } \lambda = 1, \mu > 1, \\ T_\varepsilon < \infty & \text{if } \lambda > 1 \text{ or } \lambda = 1, 0 \leq \mu \leq 1, \end{cases}$$

where  $T_\varepsilon$  is the lifespan of the smooth solution  $w = w(t, x)$  of (1.7). Therefore,  $\lambda = 1$  again appears to be the critical value for the global existence of smooth solutions  $w$  of (1.7) in the presence of the damping term  $\frac{\mu}{(1+t)^\lambda} w$ .

*Remark 1.3.* The smallness of  $\varepsilon > 0$  in Theorem 1.1 is necessary in order to guarantee the global existence of smooth solution  $(\rho, u)$ . Indeed, as in [19], large amplitude smooth solution of (1.1) may blow up in finite time even for  $0 \leq \lambda \leq 1$ . See also Theorem 4.1 in Chapter 4.

Next we concentrate on the case of  $\lambda > 1$ . As in [18], introduce the two functions

$$\begin{aligned} q_0(l) &= \int_{|x|>l} \frac{(|x| - l)^2}{|x|} (\rho(0, x) - \bar{\rho}) dx, \\ q_1(l) &= \int_{|x|>l} \frac{|x|^2 - l^2}{|x|^3} x \cdot (\rho u)(0, x) dx. \end{aligned}$$

**Theorem 1.2** (Blowup for  $\lambda > 1$ ). *Suppose  $\text{supp } \rho_0, \text{supp } u_0 \subseteq \{x : |x| \leq M\}$  and let*

$$q_0(l) > 0, \tag{1.8}$$

$$q_1(l) \geq 0 \tag{1.9}$$

*hold for all  $l \in (M_0, M)$ , where  $M_0$  is some fixed constant satisfying  $0 \leq M_0 < M$ . If  $\mu > 0$  and  $\lambda > 1$ , then there exists an  $\varepsilon_0 > 0$  such that, for  $0 < \varepsilon \leq \varepsilon_0$ , the lifespan  $T_\varepsilon$  of the smooth solution  $(\rho, u)$  of (1.1) is finite.*

*Remark 1.4.* It is not hard to find a large number of initial data  $(\rho, u)(0, \cdot)$  such that both (1.8) and (1.9) are satisfied. In addition, we would like to underline that Theorem 1.2 holds also for  $\text{curl } u_0 \neq 0$ .

Let us indicate the proofs of Theorems 1.1 and 1.2. To prove Theorem 1.1, we first introduce the function  $\psi = \frac{\varphi}{(1+t)^\lambda}$  which fulfills the second-order quasilinear wave equation

$$\partial_t^2 \psi - \Delta \psi + \frac{\mu}{(1+t)^\lambda} \partial_t \psi + \frac{2\lambda}{1+t} \partial_t \psi - \frac{\lambda(1-\lambda)}{(1+t)^2} \psi = Q(\psi),$$

where  $Q(\psi)$  stands for an error term which is of the second order in  $(\psi, \partial \psi, \partial^2 \psi)$ ;  $\partial = (\partial_t, \nabla)$ . Then, in order to establish the global existence of  $\psi$ , we introduce the time-weighted energy

$$E_N(\psi)(t) = \sum_{0 \leq |a| \leq N} \int_{\mathbb{R}^3} \left( (1+t)^{2\lambda} |\partial \Gamma^a \psi|^2 + |\Gamma^a \psi|^2 \right) dx,$$

where  $N \geq 8$  is a fixed number,  $\Gamma = (\Gamma_0, \Gamma_1, \dots, \Gamma_7) = (\partial, \Omega, S)$  with  $\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla$ ,  $S = t \partial_t + \sum_{k=1}^3 x_k \partial_k$ , and  $\Gamma^a = \Gamma_0^{a_0} \Gamma_1^{a_1} \dots \Gamma_7^{a_7}$ . Note that the vector fields  $\Gamma$  which appear in the definition of the energy  $E_N(\psi)(t)$  only comprise part of the standard Klainerman vector fields  $\{\partial, \Omega, S, H\}$ , where  $H = (H_1, H_2, H_3) = (x_1 \partial_t + t \partial_1, x_2 \partial_t + t \partial_2, x_3 \partial_t + t \partial_3)$ . This is due to the fact that the equation in (1.5) is not invariant under the Lorentz transformations  $H$  in view of the presence of the time-dependent damping. By a rather technical and involved analysis of the resulting equation for  $\psi$ , we eventually show that  $E_N(\psi)(t) \leq \frac{1}{2} K^2 \varepsilon^2$  when  $E_N(\psi)(t) \leq K^2 \varepsilon^2$  is assumed for some suitably large constant  $K > 0$

and small  $\varepsilon > 0$ . Based on this and a continuous induction argument, the global existence of  $\psi$  and then Theorem 1.1 are established for  $0 \leq \lambda \leq 1$ . To prove the blowup result of Theorem 1.2 for  $\lambda > 1$ , as in [18], we derive a related second-order ordinary differential inequality. From this and assumptions (1.8)-(1.9), an upper bound of the lifespan  $T_\varepsilon$  is derived by making essential use of  $\lambda > 1$ . In this way the proof of Theorem 1.2 is completed. Finally, in Theorem 4.1, we show that for large data smooth solution  $(\rho, u)$  of (1.1), even in case  $0 \leq \lambda \leq 1$ ,  $(\rho, u)$  will in general blow up in finite time.

Throughout, we shall use the following notation and conventions:

- $\nabla$  stands for  $\nabla_x$ ,
- $r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ ,
- $\langle r - t \rangle = (1 + (r - t)^2)^{1/2}$ ,
- $\|u(t, \cdot)\| = \left( \int_{\mathbb{R}^3} |u(t, x)|^2 dx \right)^{1/2}$  and  $\|u(t, \cdot)\|_{L^\infty} = \sup_{x \in \mathbb{R}^3} |u(t, x)|$ ,
- $Z$  denotes one of the Klainerman vector fields  $\{\partial, S, \Omega, H\}$  on  $\mathbb{R}_+ \times \mathbb{R}^3$ , where  $\partial = (\partial_t, \nabla)$ ,  $S = t\partial_t + \sum_{k=1}^3 x_k \partial_k$ ,  $\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla$ , and  $H = (H_1, H_2, H_3) = (x_1 \partial_t + t \partial_1, x_2 \partial_t + t \partial_2, x_3 \partial_t + t \partial_3)$ ,
- $\Gamma$  denotes one of the vector fields  $\{\partial, S, \Omega\}$  on  $\mathbb{R}_+ \times \mathbb{R}^3$ .
- $\beta$  is the solution of  $\beta'(t) = \frac{\mu}{(1+t)^\lambda} \beta(t)$  for  $t \geq 0$ ,  $\beta(0) = 1$ , i.e.,

$$\beta(t) \equiv \begin{cases} e^{\frac{\mu}{1-\lambda}[(1+t)^{1-\lambda}-1]}, & \lambda \geq 0, \lambda \neq 1, \\ (1+t)^\mu, & \lambda = 1. \end{cases} \quad (1.10)$$

- $c(\bar{\rho}) = 1$  will be assumed throughout (introduce  $X = x/c(\bar{\rho})$  as new space coordinate if necessary).

## 2 Global existence for small data in case $0 \leq \lambda \leq 1$

Throughout this section,  $C > 0$  stands for a generic constant which is independent of  $K, \varepsilon$ , and  $t$ .

We start by recalling the following Sobolev-type inequality (see [13]):

**Lemma 2.1.** *Let  $u = u(t, x)$  be a smooth function of  $(t, x) \in [0, \infty) \times \mathbb{R}^3$ . Then*

$$|u(t, x)| \leq C(1+r)^{-1} \sum_{|a| \leq 2} \|\Gamma^a u(t, x)\|. \quad (2.1)$$

Moreover, we shall make use of the following inequalities (see [14, Lemmas 2.3 and 3.1, Theorem 5.1]):

**Lemma 2.2.** For  $u \in C^2([0, \infty) \times \mathbb{R}^3)$ ,

$$\langle r - t \rangle \|\nabla \partial u(t, x)\| \leq C \left( \sum_{|b| \leq 1} \|\partial \Gamma^b u(t, x)\| + t \|\square u(t, x)\| \right), \quad (2.2)$$

$$(1 + r) \langle r - t \rangle |\nabla \partial u(t, x)| \leq C \left( \sum_{|b| \leq 3} \|\partial \Gamma^b u(t, x)\| + t \|\square u(t, x)\| \right), \quad (2.3)$$

where  $\square = \partial_t^2 - \Delta = \partial_t^2 - \sum_{k=1}^3 \partial_k^2$ .

We now reformulate problem (1.5). Let  $\psi = \frac{\varphi}{(1+t)^\lambda}$ . From (1.5) and  $c(\bar{\rho}) = 1$  we then have

$$\square \psi + \frac{\mu}{(1+t)^\lambda} \partial_t \psi + \frac{2\lambda}{1+t} \partial_t \psi - \frac{\lambda(1-\lambda)}{(1+t)^2} \psi = Q(\psi), \quad (2.4)$$

where

$$Q(\psi) = (c^2(\rho) - 1) \Delta \psi - 2(1+t)^\lambda \partial_t \nabla \psi \cdot \nabla \psi - 2\lambda(1+t)^{\lambda-1} |\nabla \psi|^2 \\ - \mu |\nabla \psi|^2 - (1+t)^{2\lambda} \sum_{1 \leq i, j \leq 3} (\partial_i \psi)(\partial_j \psi) \partial_{ij} \psi.$$

We define a time-weighted energy for Eq. (2.4),

$$E_N(\psi(t)) = \sum_{0 \leq |a| \leq N} \int_{\mathbb{R}^3} \left( (1+t)^{2\lambda} |\partial \Gamma^a \psi|^2 + |\Gamma^a \psi|^2 \right) dx,$$

where  $N \geq 8$  is a fixed number. Moreover, we assume that for any  $t \geq 0$

$$E_N(\psi(t)) \leq K^2 \varepsilon^2, \quad (2.5)$$

where  $K > 0$  is a suitably large constant. It follows from (2.1) and (2.5) that, for all  $|a| \leq N - 2$ ,

$$|\partial \Gamma^a \psi| \leq C(1+r)^{-1} \sum_{|b| \leq 2} \|\Gamma^b \partial \Gamma^a \psi(t, x)\| \leq C(1+r)^{-1} \sum_{|b| \leq N} \|\partial \Gamma^b \psi(t, x)\| \\ \leq C(1+r)^{-1} (1+t)^{-\lambda} \sqrt{E_N(\psi(t))} \leq CK\varepsilon(1+r)^{-1} (1+t)^{-\lambda} \quad (2.6)$$

and

$$|\Gamma^a \psi| \leq C(1+r)^{-1} \sum_{|b| \leq N} \|\Gamma^b \psi(t, x)\| \leq CK\varepsilon(1+r)^{-1}. \quad (2.7)$$

In view of Lemma 2.2 and (2.5), we have

**Lemma 2.3.** Let  $\psi$  be a solution of (2.4). Then, for all  $|a| \leq N - 3$  and  $0 \leq \lambda \leq 1$ , we have the pointwise estimate

$$\|\nabla \partial \Gamma^a \psi\|_{L^\infty} \leq CK\varepsilon(1+t)^{-2\lambda}. \quad (2.8)$$

Moreover, for  $0 \leq l \leq N - 1$ , the weighted  $L^2$  estimate

$$\begin{aligned}
& \sum_{|b| \leq l} \|\langle r-t \rangle \nabla \partial \Gamma^b \psi(t, x)\| \\
& \leq C \sum_{|c| \leq l+1} \|\partial \Gamma^c \psi(t, x)\| + C(1+t)^{1-\lambda} \sum_{|c| \leq l} \|\nabla \Gamma^c \psi(t, x)\| + C(1+t)^{-1} \sum_{|c| \leq l} \|\Gamma^c \psi\| \quad (2.9)
\end{aligned}$$

holds.

*Proof.* It follows from (2.3)–(2.4) and (2.6)–(2.7) that

$$\begin{aligned}
(1+t) \sum_{|a| \leq N-3} |\nabla \partial \Gamma^a \psi| & \leq C \sum_{|a| \leq N-3} (1+r) \langle r-t \rangle |\nabla \partial \Gamma^a \psi| \\
& \leq C \sum_{|c| \leq N} \|\partial \Gamma^c \psi\| + Ct \sum_{|a| \leq N-3} \|\square \Gamma^a \psi\| \\
& \leq CK\varepsilon(1+t)^{-\lambda} + C(1+t)^{1-\lambda} \sum_{|a| \leq N-3} \|\partial_t \Gamma^a \psi\| + C(1+t)^{-1} \sum_{|a| \leq N-3} \|\Gamma^a \psi\| \\
& \quad + C(1+t) \sum_{|b|+|c| \leq N-3} \|\nabla \partial \Gamma^b \psi \Gamma^c \psi\| + C(1+t)^{1+\lambda} \sum_{|a| \leq N-3} \|\Gamma^a (\partial_t \nabla \psi \cdot \nabla \psi)\| \\
& \leq CK\varepsilon(1+t)^{1-2\lambda} + CK\varepsilon(1+t) \sum_{|a| \leq N-3} \|\nabla \partial \Gamma^a \psi\|_{L^\infty},
\end{aligned}$$

which derives (2.7) in view of the smallness of  $\varepsilon > 0$ .

By (2.2), (2.6)-(2.8) and Eq. (2.4), we have that, for  $l \leq N-1$ ,

$$\begin{aligned}
& \sum_{|b| \leq l} \|\langle r-t \rangle \nabla \partial \Gamma^b \psi\| \\
& \leq C \sum_{|c| \leq l+1} \|\partial \Gamma^c \psi\| + Ct \sum_{|b| \leq l} \|\Gamma^b \square \psi\| \\
& \leq C \sum_{|c| \leq l+1} \|\partial \Gamma^c \psi\| + C(1+t)^{1-\lambda} \sum_{|c| \leq l} \|\nabla \Gamma^c \psi\| + C(1+t)^{-1} \sum_{|c| \leq l} \|\Gamma^c \psi\| \\
& \quad + C(1+t)^{1+\lambda} \sum_{|b| \leq l} \|\Gamma^b (\partial_t \nabla \psi \cdot \nabla \psi)\| \\
& \quad + C(1+t) \sum_{\substack{|c| \leq N-3, \\ |b| \leq l-|c|}} \|\langle r-t \rangle^{-1} \Gamma^c \psi\|_{L^\infty} \|\langle r-t \rangle \nabla \partial \Gamma^b \psi\| \\
& \quad + C(1+t) \sum_{\substack{2-N \leq |c| \leq l, \\ |b| \leq l+2-N}} \|(1+r) \nabla \partial \Gamma^b \psi\|_{L^\infty} \|(1+r)^{-1} \Gamma^c \psi\| \\
& \leq C \sum_{|c| \leq l+1} \|\partial \Gamma^c \psi\| + C(1+t)^{1-\lambda} \sum_{|c| \leq l} \|\nabla \Gamma^c \psi\| + C(1+t)^{-1} \sum_{|c| \leq l} \|\Gamma^c \psi\| \\
& \quad + CK\varepsilon \sum_{|b| \leq l} \|\langle r-t \rangle \nabla \partial \Gamma^b \psi\| + CK\varepsilon(1+t)^{1-\lambda} \sum_{2-N \leq |c| \leq l} \|(1+r)^{-1} \Gamma^c \psi\|. \quad (2.10)
\end{aligned}$$

Note that  $\Gamma^c \psi(t, x)$  is supported in  $\{x: |x| \leq t + M\}$ . Then it follows from Hardy inequality that

$$\|(1+r)^{-1} \Gamma^c \psi\| \leq C \|\nabla \Gamma^c \psi\|. \quad (2.11)$$

Substituting (2.11) into (2.10) and applying the smallness of  $\varepsilon$ , we derive (2.9).  $\square$

Next we derive the time-weighted energy estimate for the solution  $\psi$  of (2.4).

**Lemma 2.4.** *Let  $\mu > 0$  and  $\lambda \in (0, 1]$ . Under assumption (2.5), for all  $t > 0$  and  $N \geq 8$ , it holds that*

$$\begin{aligned} & \sum_{0 \leq |a| \leq N} \int_{\mathbb{R}^3} \left( (1+t)^{2\lambda} |\partial \partial^a \psi|^2 + \psi^2 \right) dx + C \sum_{0 \leq |a| \leq N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \partial^a \psi|^2 dx d\tau \\ & \leq C\varepsilon^2 + C(1+K\varepsilon) \int_0^t A(\tau) \sum_{0 \leq |a| \leq N} \int_{\mathbb{R}^3} \left( (1+\tau)^{2\lambda} |\partial \partial^a \psi|^2 + \psi^2 \right) dx d\tau, \end{aligned} \quad (2.12)$$

where  $A(\cdot)$  stands for a generic non-negative function such that  $A \in L^1((0, \infty))$  and  $\|A\|_{L^1}$  is independent of  $K$ .

*Proof.* First we show (2.12) in case  $|a| = 0$ . Multiplying Eq. (2.4) by  $m(1+t)^{2\lambda} \partial_t \psi + (1+t)^{2\lambda-1} \psi$  yields by a direct computation

$$\begin{aligned} & \frac{1}{2} \partial_t \left( m(1+t)^{2\lambda} |\partial \psi|^2 + 2(1+t)^{2\lambda-1} \psi \partial_t \psi + (\mu(1+t)^{\lambda-1} + 2\lambda(1+t)^{2\lambda-2}) \psi^2 \right) \\ & + \operatorname{div} \left( \cdots \right) + \left( \mu m(1+t)^\lambda + (\lambda m - 1)(1+t)^{2\lambda-1} \right) (\partial_t \psi)^2 + (1 - \lambda m)(1+t)^{2\lambda-1} |\nabla \psi|^2 \\ & + \mu(1-\lambda)(1+t)^{\lambda-2} \psi^2 + C_1(\lambda-1)(1+t)^{2\lambda-2} \psi \partial_t \psi + C_2(\lambda-1)(1+t)^{2\lambda-3} \psi^2 \\ & = (m(1+t)^{2\lambda} \partial_t \psi + (1+t)^\lambda \psi) Q(\psi), \end{aligned} \quad (2.13)$$

where the constant  $m > 0$  will be determined later and  $C_i$  ( $i = 1, 2$ ) are suitable constants. Note that

$$2(1+t)^{2\lambda-1} \psi \partial_t \psi \leq \sigma m(1+t)^{2\lambda} (\partial_t \psi)^2 + \frac{1}{\sigma m} (1+t)^{2\lambda-2} \psi^2 \quad (2.14)$$

for  $\sigma \in (0, 1)$ .

To guarantee the positivity of the term  $m(1+t)^{2\lambda} |\partial \psi|^2 + 2(1+t)^{2\lambda-1} \psi \partial_t \psi + (\mu(1+t)^{\lambda-1} + 2\lambda(1+t)^{2\lambda-2}) \psi^2$  in  $\partial_t(\cdot)$  and of the coefficients  $\mu m(1+t)^\lambda + (\lambda m - 1)(1+t)^{2\lambda-1}$  and  $(1 - \lambda m)(1+t)^{2\lambda-1}$  of  $(\partial_t \psi)^2$  and  $|\nabla \psi|^2$  in the left-hand side of (2.13), utilizing (2.14) with  $\sigma = 2/3$ , we may choose  $m > 0$  to fulfill

$$\lambda < \frac{1}{m} < \min\{\mu + \lambda, 2\lambda\}.$$

Then integrating (2.13) over  $\mathbb{R}^3$  yields

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \left( (1+t)^{2\lambda} |\partial \psi|^2 + (1+t)^{\lambda-1} \psi^2 \right) dx \\ & + C \int_{\mathbb{R}^3} \left( (1+t)^\lambda (\partial_t \psi)^2 + (1+t)^{2\lambda-1} |\nabla \psi|^2 + (1+t)^{\lambda-2} \psi^2 \right) dx \end{aligned}$$



$$\leq A(t) \int_{\mathbb{R}^3} (1+t)^{\lambda-1} \psi^2 dx + C \left| \int_{\mathbb{R}^3} (m(1+t)^{2\lambda} \partial_t \psi + (1+t)^{2\lambda-1} \psi) Q(\psi) dx \right|. \quad (2.15)$$

Next we improve the time-weighted estimate of  $\psi$  in the left-hand side of (2.15). Multiplying both sides of (2.4) by  $(1+t)^\lambda \psi$  yields by direct computation

$$\begin{aligned} & \partial_t \left( (1+t)^\lambda \psi \partial_t \psi + \frac{\mu}{2} \psi^2 \right) + \operatorname{div}(\dots) - (1+t)^\lambda (\partial_t \psi)^2 - \lambda(1+t)^{\lambda-1} \psi \partial_t \psi \\ & + (1+t)^\lambda |\nabla \psi|^2 + 2\lambda(1+t)^{\lambda-1} \psi \partial_t \psi + \lambda(\lambda-1)(1+t)^{\lambda-2} \psi^2 = (1+t)^\lambda \psi Q(\psi). \end{aligned}$$

From this and (2.15), we can choose the multiplier  $m(1+t)^{2\lambda} \partial_t \psi + (1+t)^{2\lambda-1} \psi + \kappa(1+t)^\lambda \psi$  for Eq. (2.4) with a small  $\kappa > 0$  and then obtain by an integration with respect to the time variable  $t$

$$\begin{aligned} & \int_{\mathbb{R}^3} \left( (1+t)^{2\lambda} |\partial \psi|^2 + \psi^2 \right) dx + C \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \psi|^2 dx d\tau \\ & \leq C\varepsilon^2 + \int_0^t A(\tau) \int_{\mathbb{R}^3} \psi^2 dx d\tau + C \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \psi) Q(\psi) dx \right| d\tau \\ & \quad + C \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^\lambda \psi Q(\psi) dx \right| d\tau. \quad (2.16) \end{aligned}$$

Next we derive the time-weighted estimates of  $\partial^\alpha \psi$  with  $1 \leq |\alpha| \leq N$ . Taking  $\partial^\alpha$  on both sides of Eq. (2.4) yields

$$\begin{aligned} & \square \partial^a \psi + \frac{\mu}{(1+t)^\lambda} \partial_t \partial^a \psi + \frac{2\lambda}{1+t} \partial_t \partial^a \psi \\ & = \partial^a Q(\psi) + \sum_{1 \leq |b| \leq |\alpha|} \frac{1}{(1+t)^\lambda} \left( 1 + O((1+t)^{\lambda-1}) \right) \partial^b \psi - \lambda(\lambda-1) \partial^a \left( \frac{1}{(1+t)^2} \right) \psi. \end{aligned}$$

Exactly as for (2.16), we obtain

$$\begin{aligned} & \sum_{0 \leq |a| \leq N} \int_{\mathbb{R}^3} \left( (1+t)^{2\lambda} |\partial^{a+1} \psi|^2 + \psi^2 \right) dx + C \sum_{0 \leq |a| \leq N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial^{a+1} \psi|^2 dx d\tau \\ & \leq C\varepsilon^2 + \int_0^t A(\tau) \int_{\mathbb{R}^3} \psi^2 dx d\tau + C \sum_{0 \leq |a| \leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \partial^a \psi) \partial^a Q(\psi) dx \right| d\tau \\ & \quad + C \sum_{0 \leq |a| \leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^\lambda (\partial^a \psi) \partial^a Q(\psi) dx \right| d\tau. \quad (2.17) \end{aligned}$$

We now deal with the last two terms in the right-hand side of (2.17). We first analyze the integrand  $(1+t)^{2\lambda} (\partial_t \partial^a \psi) \partial^a Q(\psi)$  of  $\int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \partial^a \psi) \partial^a Q(\psi) dx \right| d\tau$ . Direct computation yields

$$\partial^a Q(\psi) = (c^2(\rho) - 1) \Delta \partial^a \psi - 2(1+t)^\lambda \nabla \partial_t \partial^a \psi \cdot \nabla \psi - (1+t)^{2\lambda} (\partial_i \psi) (\partial_j \psi) \partial_{ij} \partial^a \psi + \text{l.o.t.}$$

and

$$\begin{aligned}
(1+t)^{2\lambda}(\partial_t \partial^a \psi) \partial^a Q(\psi) &= \operatorname{div} \left( (1+t)^{2\lambda}(c^2(\rho)-1)(\partial_t \partial^a \psi) \nabla \partial^a \psi \right) - \operatorname{div} \left( (1+t)^{3\lambda} |\partial_t \partial^a \psi|^2 \nabla \psi \right) \\
&- \frac{1}{2} \partial_t \left( (1+t)^{2\lambda}(c^2(\rho)-1) |\nabla \partial^a \psi|^2 \right) + (1+t)^{3\lambda} |\partial_t \partial^a \psi|^2 \Delta \psi + \lambda(1+t)^{2\lambda-1}(c^2(\rho)-1) |\nabla \partial^a \psi|^2 \\
&+ \frac{1}{2} (1+t)^{2\lambda}(c^2(\rho))' \partial_t \rho |\nabla \partial^a \psi|^2 - (1+t)^{4\lambda} (\partial_i \psi) (\partial_j \psi) (\partial_{ij} \partial^a \psi) \partial_t \partial^a \psi + \text{l.o.t.}, \quad (2.18)
\end{aligned}$$

where here and below l.o.t. designates lower-order terms which are of the form  $(\partial^{b_1} \psi)(\partial^{b_2} \psi) \dots (\partial^{b_l} \psi)$  (multiplied by  $\partial \partial^a \psi$  or  $\partial^a \psi$ ) with  $l \geq 2$  and  $1 \leq |b_1| + \dots + |b_l| \leq |a| + 1$ . Here we are concerned with the top-order derivatives only. Note that the term  $(1+t)^{4\lambda} (\partial_i \psi) (\partial_j \psi) (\partial_{ij} \partial^a \psi) \partial_t \partial^a \psi$  in (2.18) can be expressed as

$$\begin{aligned}
&(1+t)^{4\lambda} (\partial_i \psi) (\partial_j \psi) (\partial_{ij} \partial^a \psi) \partial_t \partial^a \psi \\
&= \frac{1}{2} \left\{ \partial_i \left( (1+t)^{4\lambda} (\partial_i \psi) (\partial_j \psi) (\partial_j \partial^a \psi) \partial_t \partial^a \psi \right) + \partial_j \left( (1+t)^{4\lambda} (\partial_i \psi) (\partial_j \psi) (\partial_i \partial^a \psi) \partial_t \partial^a \psi \right) \right. \\
&\quad \left. - \partial_i \left( (1+t)^{4\lambda} (\partial_i \psi) (\partial_j \psi) (\partial_i \partial^a \psi) \partial_j \partial^a \psi \right) + \partial_t \left( (1+t)^{4\lambda} (\partial_i \psi) \partial_j \psi (\partial_i \partial^a \psi) \partial_j \partial^a \psi + \text{l.o.t.} \right) \right\}. \quad (2.19)
\end{aligned}$$

Similarly, for the integrand  $(1+t)^\lambda (\partial^a \psi) \partial^a Q(\psi)$  of  $\int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^\lambda (\partial^a \psi) \partial^a Q(\psi) dx \right| d\tau$ , one has

$$\begin{aligned}
&(1+t)^\lambda \partial^a \psi \partial^a Q(\psi) \\
&= \operatorname{div} \left( (1+t)^\lambda (c^2(\rho)-1) \nabla (\partial^a \psi) \partial^a \psi \right) - \frac{1}{2} \partial_i \left( (1+t)^{3\lambda} (\partial_i \psi) \partial^a (|\nabla \psi|^2) \partial^a \psi \right) \\
&\quad - \partial_t \left( (1+t)^\lambda \partial^a (|\nabla \psi|^2) \partial^a \psi \right) - (1+t)^\lambda (c^2(\rho)-1) |\nabla \partial^a \psi|^2 \\
&\quad - (1+t)^\lambda (c^2(\rho))' \nabla \rho \cdot \nabla (\partial^a \psi) \partial^a \psi + \lambda(1+t)^{\lambda-1} \partial^a (|\nabla \psi|^2) \partial^a \psi \\
&\quad + (1+t)^\lambda \partial^a (|\nabla \psi|^2) \partial_t \partial^a \psi + \frac{1}{2} (1+t)^{3\lambda} (\Delta \psi) \partial^a (|\nabla \psi|^2) \partial^a \psi \\
&\quad + \frac{1}{2} (1+t)^{3\lambda} \nabla \psi \cdot \nabla (\partial^a \psi) \partial^a (|\nabla \psi|^2) + \text{l.o.t.} \quad (2.20)
\end{aligned}$$

From the expression  $(\partial^{b_1} \psi)(\partial^{b_2} \psi) \dots (\partial^{b_l} \psi)$  ( $l \geq 2$ ,  $1 \leq |b_1| + \dots + |b_l| \leq N+1$ ) of the lower-order terms one readily obtains that there exists at most one  $b_j$  ( $1 \leq j \leq l$ ) such that  $\left\lceil \frac{N+3}{2} \right\rceil < |b_j| \leq N+1$ .

Moreover,  $\left\lceil \frac{N+3}{2} \right\rceil \leq N-2$  by  $N \geq 8$ . Thus, applying (2.5)-(2.7) and subsequently substituting (2.18)-(2.20) into (2.17), completes the proof of Lemma 2.4.  $\square$

Next we focus on the general time-weighted energy estimate of  $\partial \Gamma^a \psi$  with  $0 \leq |a| \leq N$  and  $N \geq 8$ .

**Lemma 2.5** (Time-weighted energy estimate of  $\partial \Gamma^a \psi$  for  $|a| \leq N$ ). *Let  $\mu > 0$  and  $\lambda \in (0, 1]$ . Under assumption (2.5), we have that, for  $t > 0$ ,*

$$\sum_{0 \leq |a| \leq N} \int_{\mathbb{R}^3} \left( (1+t)^{2\lambda} |\partial \Gamma^a \psi|^2 + |\Gamma^a \psi|^2 \right) dx + C \sum_{0 \leq |\alpha| \leq N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \Gamma^a \psi|^2 dx d\tau$$

$$\leq C\varepsilon^2 + C(1 + K\varepsilon) \int_0^t A(\tau) \sum_{0 \leq |a| \leq N} \int_{\mathbb{R}^3} \left( (1 + \tau)^{2\lambda} |\partial \Gamma^a \psi|^2 + \psi^2 \right) dx d\tau, \quad (2.21)$$

where the function  $A$  has been defined in Lemma 2.4.

*Proof.* Writing  $\Gamma^a = \tilde{\Gamma}^b \partial^c$  with  $\tilde{\Gamma} \in \{\Omega, S\}$ , we will use induction on  $|b|$  to prove (2.21). In view of Lemma 2.4, it is enough to assume that  $|c| = 0$ .

Suppose that (2.21) holds for  $|b| \leq l - 1$ , where  $1 \leq l \leq N$ . We then intend to establish (2.21) for  $|b| = l$ .

Acting with  $\tilde{\Gamma}^a$  (where  $a = b$  and  $|b| = l$ ) on both sides of (2.4) yields

$$\begin{aligned} \square \tilde{\Gamma}^a \psi + \frac{\mu}{(1+t)^\lambda} \partial_t \tilde{\Gamma}^a \psi + \frac{2\lambda}{1+t} \partial_t \tilde{\Gamma}^a \psi &= \sum_{|b_1| < |b|} \tilde{\Gamma}^{b_1} \partial^c \square \psi \\ &+ \tilde{\Gamma}^a Q(\psi) - \left[ \tilde{\Gamma}^a, \frac{\mu}{(1+t)^\lambda} \partial_t \right] \psi - \left[ \tilde{\Gamma}^a, \frac{2\lambda}{1+t} \partial_t \right] \psi + \tilde{\Gamma}^a ((\lambda - 1)(1+t)^{-2} \psi). \end{aligned} \quad (2.22)$$

Starting from (2.22), as in the proof of Lemma 2.4, we can choose the multiplier  $m(1+t)^{2\lambda} \partial_t \tilde{\Gamma}^a \psi + (1+t)^{2\lambda-1} \tilde{\Gamma}^a \psi + \kappa(1+t)^\lambda \tilde{\Gamma}^a \psi$  to derive (2.21). Indeed, it follows from a direct computation that

$$\begin{aligned} &\sum_{\substack{|b|=l, \\ |c| \leq N-l}} \int_{\mathbb{R}^3} \left( (1+t)^{2\lambda} |\partial \tilde{\Gamma}^b \partial^c \psi|^2 + |\tilde{\Gamma}^b \partial^c \psi|^2 \right) dx + C \sum_{\substack{|b|=l, \\ |c| \leq N-l}} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \tilde{\Gamma}^b \partial^c \psi|^2 dx d\tau \\ &\leq C\varepsilon^2 + C \sum_{\substack{|b_1| < l, \\ |c_1| \leq N-|b_1|}} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \tilde{\Gamma}^{b_1} \partial^{c_1} \psi|^2 dx d\tau \\ &+ C(1 + K\varepsilon) \int_0^t A(\tau) \sum_{\substack{|b_1| \leq l, \\ |c_1| \leq N-|b_1|}} \int_{\mathbb{R}^3} \left( (1+\tau)^{2\lambda} |\partial \tilde{\Gamma}^{b_1} \partial^{c_1} \psi|^2 + |\tilde{\Gamma}^{b_1} \partial^{c_1} \psi|^2 \right) dx d\tau \\ &+ C \sum_{\substack{|b_1| \leq l, \\ |c_1| \leq N-|b_1|}} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1} \partial^{c_1} Q(\psi) dx \right| d\tau \\ &+ C \sum_{\substack{|b_1| \leq l, \\ |c_1| \leq N-|b_1|}} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^\lambda (\tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1} \partial^{c_1} Q(\psi) dx \right| d\tau. \end{aligned} \quad (2.23)$$

Next we deal with the last two terms in the right-hand side of (2.23). Note that

$$c^2(\rho) - 1 = -G(\psi) \int_0^1 (c^2)'(-sG(\psi)) ds,$$

where  $G(\psi) = (1+t)^\lambda \partial_t \psi + (1+t)^{\lambda-1} \psi + (1+t)^{2\lambda} |\nabla \psi|^2 / 2 + \mu \psi$ . From this, it is readily seen that the typical terms in  $Q(\psi)$  are of the form  $\psi \Delta \psi$ ,  $(1+t)^\lambda \partial_t \nabla \psi \cdot \nabla \psi$ , and  $(1+t)^{2\lambda} (\partial_i \psi) (\partial_j \psi) \partial_{ij} \psi$ . We analyze them separately. Without loss of generality, we assume  $|c_1| = 0$  in the last two terms of (2.23); the treatment of the other cases is easier.

**Part A** Estimates of  $\sum_{|b_1| \leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1} (\psi \Delta \psi) dx \right| d\tau$  and  $\sum_{|b_1| \leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^\lambda (\tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1} (\psi \Delta \psi) dx \right| d\tau$ .

Note that

$$\tilde{\Gamma}^{b_1}(\psi \Delta \psi) = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3,$$

where

$$\begin{aligned} \mathbf{I}_1 &= \psi \Delta \tilde{\Gamma}^{b_1} \psi, \\ \mathbf{I}_2 &= \sum_{\substack{|b_1|=|b_2|+|b_3|, \\ 1 \leq |b_2| \leq N-3}} (\tilde{\Gamma}^{b_2} \psi) \Delta \tilde{\Gamma}^{b_3} \psi, \\ \mathbf{I}_3 &= \sum_{\substack{|b_1|=|b_2|+|b_3|, \\ N-2 \leq |b_2| \leq l}} (\tilde{\Gamma}^{b_2} \psi) \Delta \tilde{\Gamma}^{b_3} \psi. \end{aligned}$$

In view of  $b_1 = a$  and

$$\begin{aligned} & (1+t)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \psi \Delta \tilde{\Gamma}^a \psi \\ &= \operatorname{div} \left( (1+t)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \psi \nabla \tilde{\Gamma}^a \psi \right) + \frac{1}{2} \partial_t \left( (1+t)^{2\lambda} |\nabla \tilde{\Gamma}^a \psi|^2 \psi \right) \\ & - (1+t)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \nabla \psi \cdot \nabla \tilde{\Gamma}^a \psi - \lambda (1+t)^{\lambda-1} |\nabla \tilde{\Gamma}^a \psi|^2 \psi - \frac{1}{2} (1+t)^{2\lambda} |\nabla \tilde{\Gamma}^a \psi|^2 \partial_t \psi, \end{aligned}$$

we have by an integration by parts and (2.6)-(2.7)

$$\begin{aligned} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \mathbf{I}_1 dx \right| d\tau &\leq C\varepsilon^2 + CK\varepsilon \sum_{0 \leq |a| \leq N} \int_{\mathbb{R}^3} (1+t)^{2\lambda} |\partial \tilde{\Gamma}^a \psi|^2 dx \\ &+ CK\varepsilon \sum_{0 \leq |a| \leq N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \tilde{\Gamma}^a \psi|^2 dx d\tau. \end{aligned} \quad (2.24)$$

Moreover, it follows from (2.7) and (2.9) that

$$\begin{aligned} & \int_{\mathbb{R}^3} \left| (1+t)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) (\tilde{\Gamma}^{b_2} \psi) \Delta \tilde{\Gamma}^{b_3} \psi \right| dx \\ & \leq (1+t)^{2\lambda} \|\langle r-t \rangle^{-1} \tilde{\Gamma}^{b_2} \psi\|_{L^\infty} \cdot \|\partial_t \tilde{\Gamma}^a \psi\| \cdot \|\langle r-t \rangle \Delta \tilde{\Gamma}^{b_3} \psi\| \\ & \leq CK\varepsilon (1+t)^\lambda \|\partial_t \tilde{\Gamma}^a \psi\| \sum_{|b_4| \leq |b_3|+1} \left( \|\nabla \tilde{\Gamma}^{b_4} \psi\| + (1-\lambda)(1+t)^{-1} \|\tilde{\Gamma}^{b_4} \psi\| \right) \\ & \leq CK\varepsilon (1+t)^\lambda \|\partial_t \tilde{\Gamma}^a \psi\| \sum_{|b_4| \leq |b_3|+1} \|\nabla \tilde{\Gamma}^{b_4} \psi\| + CK\varepsilon (1+t)^\lambda \|\partial_t \tilde{\Gamma}^a \psi\|^2 \\ & \quad + CK\varepsilon (1-\lambda)(1+t)^{\lambda-2} \sum_{|b_4| \leq |b_3|+1} \|\tilde{\Gamma}^{b_4} \psi\|^2. \end{aligned} \quad (2.25)$$

On the other hand, we have that by (2.6) and Hardy's inequality

$$\begin{aligned}
& \int_{\mathbb{R}^3} \left| (1+t)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) (\Gamma^{b_2} \psi) \Delta \tilde{\Gamma}^{b_3} \psi \right| dx \\
& \leq (1+t)^{2\lambda} \|(1+r) \Delta \tilde{\Gamma}^{b_3} \psi\|_{L^\infty} \cdot \|\partial_t \tilde{\Gamma}^a \psi\| \|(1+r)^{-1} \tilde{\Gamma}^{b_2} \psi\| \\
& \leq CK\varepsilon (1+t)^\lambda \|\partial_t \tilde{\Gamma}^a \psi\| \sum_{|b_4| \leq |b_2|} \|\nabla \tilde{\Gamma}^{b_4} \psi\|. \tag{2.26}
\end{aligned}$$

Combining (2.24)-(2.26) yields

$$\begin{aligned}
& \sum_{|b_1| \leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \Gamma^a \psi) \Gamma^{b_1} (\psi \Delta \psi) dx \right| d\tau \\
& \leq C\varepsilon^2 + CK\varepsilon \sum_{0 \leq |a| \leq N} \int_{\mathbb{R}^3} (1+t)^{2\lambda} |\partial \tilde{\Gamma}^a \psi|^2 dx + CK\varepsilon \sum_{0 \leq |a| \leq N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \tilde{\Gamma}^a \psi|^2 dx d\tau \\
& \quad + CK\varepsilon \sum_{|b_4| \leq N} \int_0^t A(\tau) \int_{\mathbb{R}^3} |\tilde{\Gamma}^{b_4} \psi|^2 dx d\tau. \tag{2.27}
\end{aligned}$$

Note that

$$\begin{aligned}
(1+t)^\lambda (\tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1} (\psi \Delta \psi) &= \sum_{|b_2|+|b_3|=|b_1|} (1+t)^\lambda (\tilde{\Gamma}^a \psi) (\tilde{\Gamma}^{b_2} \psi) \Delta \tilde{\Gamma}^{b_3} \psi \\
&= \operatorname{div} \left( \sum_{|b_2|+|b_3|=|b_1|} (1+t)^\lambda (\tilde{\Gamma}^a \psi) (\tilde{\Gamma}^{b_2} \psi) \nabla \tilde{\Gamma}^{b_3} \psi \right) + \sum_{i=4}^5 \mathbf{I}_i,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{I}_4 &= - \sum_{\substack{|b_2| \leq N-3, \\ |b_2|+|b_3|=|b_1|}} (1+t)^\lambda (\tilde{\Gamma}^{b_2} \psi) (\nabla \tilde{\Gamma}^a \psi) \cdot (\nabla \tilde{\Gamma}^{b_3} \psi), \\
\mathbf{I}_5 &= - \sum_{\substack{N-2 \leq |b_2| \leq l-1, \\ |b_2|+|b_3|=|b_1|}} (1+t)^\lambda (\tilde{\Gamma}^{b_2} \psi) (\nabla \tilde{\Gamma}^a \psi) \cdot (\nabla \tilde{\Gamma}^{b_3} \psi) \\
&\quad - \sum_{|b_2|+|b_3|=|b_1|} (1+t)^\lambda (\tilde{\Gamma}^a \psi) (\nabla \tilde{\Gamma}^{b_2} \psi) \cdot (\nabla \tilde{\Gamma}^{b_3} \psi).
\end{aligned}$$

Therefore, by (2.7) and Hardy's inequality, we have

$$\int_{\mathbb{R}^3} |\mathbf{I}_4| dx \leq CK\varepsilon (1+t)^\lambda \|\nabla \tilde{\Gamma}^a \psi\| \sum_{|b_1|+3-N \leq |b_3| \leq N} \|\nabla \tilde{\Gamma}^{b_3} \psi\|$$

and

$$\int_{\mathbb{R}^3} |\mathbf{I}_5| dx \leq CK\varepsilon \|(1+r)^{-1} \tilde{\Gamma}^{b_2} \psi \nabla \tilde{\Gamma}^a \psi\|_{L^1} \leq CK\varepsilon \|\nabla \tilde{\Gamma}^{b_2} \psi\| \|\nabla \tilde{\Gamma}^a \psi\|.$$

This yields

$$\sum_{|b_1| \leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^\lambda (\tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1} (\psi \Delta \psi) dx \right| d\tau \leq CK\varepsilon \sum_{0 \leq |a| \leq N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \tilde{\Gamma}^a \psi|^2 dx d\tau. \tag{2.28}$$

**Part B** Estimates of  $\sum_{|b_1| \leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1} \left( (1+\tau)^\lambda \partial_t \nabla \psi \cdot \nabla \psi \right) dx \right| d\tau$  and  $\sum_{|b_1| \leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^\lambda (\tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1} \left( (1+\tau)^\lambda \partial_t \nabla \psi \cdot \nabla \psi \right) dx \right| d\tau$ .

One has

$$\begin{aligned} \tilde{\Gamma}^{b_1} \left( (1+t)^\lambda \partial_t \nabla \psi \cdot \nabla \psi \right) &= (1+t)^\lambda \partial_t \nabla \tilde{\Gamma}^{b_1} \psi \cdot \nabla \psi + \sum_{N-3 \leq |b_2| \leq l-1} (1+t)^\lambda (\partial_t \nabla \tilde{\Gamma}^{b_2} \psi) \nabla \tilde{\Gamma}^{b_3} \psi \\ &\quad + \sum_{|b_2| \leq N-4} (1+t)^\lambda (\partial_t \nabla \tilde{\Gamma}^{b_2} \psi) \nabla \tilde{\Gamma}^{b_3} \psi \\ &= \Pi_1 + \Pi_2 + \Pi_3. \end{aligned}$$

By (2.6), we have

$$\sum_{|b_1| \leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \Pi_1 dx \right| d\tau \leq CK\varepsilon \sum_{0 \leq |a| \leq N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \tilde{\Gamma}^a \psi|^2 dx d\tau. \quad (2.29)$$

In addition, it follows from (2.6), (2.8) and a direct computation that

$$\begin{aligned} (1+t)^{2\lambda} \|(\partial_t \Gamma^a \psi) \Pi_2\|_{L^1} &\leq (1+t)^{3\lambda} \sum_{|b_2| \leq N-4} \|\langle r-t \rangle^{-1} \nabla \Gamma^{b_3} \psi\|_{L^\infty} \cdot \|\partial_t \Gamma^a \psi\| \cdot \|\langle r-t \rangle \partial_t \nabla \Gamma^{b_2} \psi\| \\ &\leq CK\varepsilon (1+t)^\lambda \|\partial_t \Gamma^a \psi\| \sum_{|c| \leq |b_2|+1} (\|\nabla \Gamma^c \psi\| + (1-\lambda)(1+t)^{-1} \|\Gamma^c \psi\|) \\ &\leq CK\varepsilon (1+t)^\lambda \|\partial_t \tilde{\Gamma}^a \psi\| \sum_{|b_4| \leq |b_3|+1} \|\nabla \tilde{\Gamma}^{b_4} \psi\| + CK\varepsilon (1+t)^\lambda \|\partial_t \tilde{\Gamma}^a \psi\|^2 \\ &\quad + CK\varepsilon (1-\lambda)(1+t)^{\lambda-2} \sum_{|b_4| \leq |b_3|+1} \|\tilde{\Gamma}^{b_4} \psi\|^2. \end{aligned} \quad (2.30)$$

Treating  $\Pi_3$ , we obtain by (2.7)

$$\int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \Pi_3 dx \right| d\tau \leq CK\varepsilon \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \tilde{\Gamma}^a \psi|^2 dx d\tau. \quad (2.31)$$

Collecting (2.29)-(2.31) yields

$$\begin{aligned} &\sum_{|b_1| \leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \Gamma^a \psi) \Gamma^{b_1} \left( (1+t)^\lambda \partial_t \nabla \psi \cdot \nabla \psi \right) dx \right| d\tau \\ &\leq CK\varepsilon \sum_{0 \leq |a| \leq N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \tilde{\Gamma}^a \psi|^2 dx d\tau + CK\varepsilon \sum_{|b_4| \leq N} \int_0^t A(\tau) \int_{\mathbb{R}^3} |\tilde{\Gamma}^{b_4} \psi|^2 dx d\tau. \end{aligned} \quad (2.32)$$

In addition, one notes that

$$2(1+t)^{2\lambda}(\tilde{\Gamma}^a\psi)\tilde{\Gamma}^a(\partial_t\nabla\psi\cdot\nabla\psi) = \sum_{|c|\leq|a|} \partial_t\left((1+t)^{2\lambda}\tilde{\Gamma}^a\psi\Gamma^c(|\nabla\psi|^2)\right) \\ - 2\lambda(1+t)^{2\lambda-1}(\tilde{\Gamma}^a\psi)\tilde{\Gamma}^c(|\nabla\psi|^2) - (1+t)^{2\lambda}(\partial_t\tilde{\Gamma}^a\psi)\tilde{\Gamma}^c(|\nabla\psi|^2).$$

From this and (2.7), we have

$$\sum_{|b_1|\leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^\lambda (\tilde{\Gamma}^a\psi) \tilde{\Gamma}^{b_1} \left( (1+\tau)^\lambda \partial_t \nabla \psi \cdot \nabla \psi \right) dx \right| d\tau \leq C\varepsilon^2 \\ + CK\varepsilon \sum_{0\leq|a|\leq N} \int_{\mathbb{R}^3} (1+t)^{2\lambda} |\partial \tilde{\Gamma}^a \psi|^2 dx + CK\varepsilon \sum_{0\leq|a|\leq N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \tilde{\Gamma}^a \psi|^2 dx d\tau. \quad (2.33)$$

**Part C** Estimates of  $\sum_{|b_1|\leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1} \left( (1+\tau)^{2\lambda} (\partial_i \psi) (\partial_j \psi) \partial_{ij} \psi \right) dx \right| d\tau$  and  $\sum_{|b_1|\leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^\lambda (\tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1} \left( (1+\tau)^{2\lambda} (\partial_i \psi) (\partial_j \psi) \partial_{ij} \psi \right) dx \right| d\tau$ .

A direct computation yields

$$\tilde{\Gamma}^{b_1}((\partial_i \psi)(\partial_j \psi) \partial_{ij} \psi) = \frac{1}{2} \tilde{\Gamma}^{b_1}((\partial_i \psi) \partial_i (|\nabla \psi|^2)) \\ = \frac{1}{2} (\partial_i \psi) \partial_i \tilde{\Gamma}^{b_1}(|\nabla \psi|^2) + \sum_{N-3\leq|b_2|\leq|b_1|-1} (\nabla^2 \tilde{\Gamma}^{b_2} \psi)(\nabla \tilde{\Gamma}^{b_3} \psi) \nabla \tilde{\Gamma}^{b_4} \psi \\ + \sum_{|b_2|\leq N-4} (\nabla^2 \tilde{\Gamma}^{b_2} \psi)(\nabla \tilde{\Gamma}^{b_3} \psi) \nabla \tilde{\Gamma}^{b_4} \psi \\ = \text{III}_1 + \text{III}_2 + \text{III}_3.$$

As in the treatment of  $\text{II}_1$  in Part B, we have

$$\sum_{|b_1|\leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \text{III}_1 dx \right| d\tau \leq CK\varepsilon \sum_{0\leq|a|\leq N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \tilde{\Gamma}^a \psi|^2 dx d\tau. \quad (2.34)$$

By (2.6) and (2.9), for the term  $\text{III}_2$ , we have

$$(1+t)^{4\lambda} \|(\partial_t \tilde{\Gamma}^a \psi)(\nabla^2 \tilde{\Gamma}^{b_2} \psi)(\nabla \tilde{\Gamma}^{b_3} \psi) \nabla \tilde{\Gamma}^{b_4} \psi\|_{L^1} \\ \leq (1+t)^{4\lambda} \|\langle r-t \rangle^{-1} (\nabla \tilde{\Gamma}^{b_3} \psi) \nabla \tilde{\Gamma}^{b_4} \psi\|_{L^\infty} \cdot \|\partial_t \tilde{\Gamma}^a \psi\| \cdot \|\langle r-t \rangle \nabla^2 \tilde{\Gamma}^{b_2} \psi\| \\ \leq CK\varepsilon(1+t)^\lambda \|\partial_t \tilde{\Gamma}^a \psi\| \sum_{|b_4|\leq|b_3|+1} \|\nabla \tilde{\Gamma}^{b_4} \psi\| + CK\varepsilon(1+t)^\lambda \|\partial_t \tilde{\Gamma}^a \psi\|^2 \\ + CK\varepsilon(1-\lambda)(1+t)^{\lambda-2} \sum_{|b_4|\leq|b_3|+1} \|\tilde{\Gamma}^{b_4} \psi\|^2. \quad (2.35)$$

By (2.7) and (2.8), for the term  $\text{III}_3$ , one has

$$(1+t)^{4\lambda} \|(\partial_t \tilde{\Gamma}^a \psi)(\nabla^2 \tilde{\Gamma}^{b_2} \psi)(\nabla \tilde{\Gamma}^{b_3} \psi) \nabla \tilde{\Gamma}^{b_4} \psi\|_{L^1} \leq CK\varepsilon(1+t)^\lambda \|\partial_t \tilde{\Gamma}^a \psi\| \sum_{|c|\leq|b_1|} \|\nabla \tilde{\Gamma}^c \psi\|. \quad (2.36)$$

Collecting (2.34)-(2.36) yields

$$\begin{aligned} & \sum_{|b_1| \leq N} \int_0^t \left| \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1} \left( (1+\tau)^{2\lambda} (\partial_i \psi) (\partial_j \psi) \partial_{ij} \psi \right) dx \right| d\tau \\ & \leq CK\varepsilon \sum_{0 \leq |a| \leq N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \tilde{\Gamma}^a \psi|^2 dx d\tau + CK\varepsilon \sum_{|b_4| \leq N} \int_0^t A(\tau) \int_{\mathbb{R}^3} |\tilde{\Gamma}^{b_4} \psi|^2 dx d\tau. \end{aligned} \quad (2.37)$$

In addition,

$$\begin{aligned} & 2(1+t)^{3\lambda} (\Gamma^a \psi) \Gamma^{b_1} ((\partial_i \psi) (\partial_j \psi) \partial_{ij} \psi) \\ & = \operatorname{div} \left( (1+t)^{3\lambda} (\Gamma^a \psi) (\nabla \psi) \Gamma^{b_1} (|\nabla \psi|^2) \right) - (1+t)^{3\lambda} (\nabla \Gamma^a \psi) (\nabla \psi) \Gamma^{b_1} (|\nabla \psi|^2) \\ & - (1+t)^{3\lambda} (\Gamma^a \psi) (\Delta \psi) \Gamma^{b_1} (|\nabla \psi|^2) + \sum_{|b_2| \leq |b_1| - 1} (1+t)^{3\lambda} (\Gamma^a \psi) (\nabla^2 \Gamma^{b_2} \psi) (\nabla \Gamma^{b_3} \psi) \nabla \Gamma^{b_4} (|\psi|^2). \end{aligned}$$

Together with (2.6)-(2.7) this yields

$$\begin{aligned} & \sum_{|b_1| \leq N} \left| \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda (\Gamma^a \psi) \Gamma^{b_1} \left( (1+\tau)^{2\lambda} (\partial_i \psi) (\partial_j \psi) \partial_{ij} \psi \right) dx d\tau \right| \\ & \leq CK\varepsilon \sum_{|a| \leq N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^\lambda |\partial \Gamma^a \psi|^2 dx d\tau. \end{aligned} \quad (2.38)$$

Therefore, substituting (2.27), (2.28), (2.32)-(2.33), and (2.37)-(2.38) into (2.23) and utilizing the smallness of  $\varepsilon > 0$  gives (2.21).  $\square$

Based on Lemmas 2.4–2.5, we now prove Theorem 1.1.

*Proof of Theorem 1.1.* By Lemma 2.4 and Lemma 2.5, one has that, for fixed  $N \geq 9$ ,

$$E_N(t) \leq C\varepsilon^2 + C(1+K\varepsilon) \int_0^t A(t') E_N(t') dt'.$$

Choosing the constants  $K > 0$  large and  $\varepsilon > 0$  small, by Gronwall's inequality one gets that, for any  $t \geq 0$ ,

$$E_N(t) \leq e^{C(1+K\varepsilon)\|A(t)\|_{L^1}} E_N(0) \leq \frac{1}{2} K^2 \varepsilon^2$$

Thus, Theorem 1.1 is proved by the assumption that  $E_N(t) \leq K^2 \varepsilon^2$  and a continuous induction argument.  $\square$

### 3 Blowup for small data in case $\lambda > 1$

In this section, we shall prove the blowup result of Theorem 1.2 which is valid in case  $\lambda > 1$ .

*Proof of Theorem 1.2.* We divide the proof into two cases.



**Case 1:  $\gamma = 2$ .**

Let  $(\rho, u)$  be a smooth solution of (1.1). For  $l > 0$ , we define

$$P(t, l) = \int_{|x|>l} \eta(x, l) (\rho(t, x) - \bar{\rho}) dx, \quad (3.1)$$

where

$$\eta(x, l) = |x|^{-1}(|x| - l)^2.$$

Employing the first equation in (1.1) and an integration by parts, we see that

$$\begin{aligned} \partial_t P(t, l) &= \int_{|x|>l} \eta(x, l) \partial_t (\rho(t, x) - \bar{\rho}) dx = - \int_{|x|>l} \eta(x, l) \operatorname{div}(\rho u)(t, x) dx \\ &= \int_{|x|>l} (\nabla_x \eta)(x, l) \cdot (\rho u)(t, x) dx, \end{aligned}$$

where we have used the fact that  $\eta(x, l) = 0$  on  $|x| = l$  and that  $u(t, x) = 0$  for  $|x| \geq t + M$ .

We first show that, under the assumptions (1.8)-(1.9),  $P(t, l)$  defined by (3.1) is nonnegative for  $l \geq M_0$ . By differentiating  $\partial_t P(t, l)$  again and using the second equation in (1.1), we find that

$$\begin{aligned} \partial_t^2 P(t, l) &= \int_{|x|>l} (\nabla_x \eta)(x, l) \cdot \partial_t (\rho u)(t, x) dx = - \sum_{i,j} \int_{|x|>l} (\partial_{x_i} \eta) \partial_{x_j} (\rho u_i u_j) dx \\ &\quad - \int_{|x|>l} (\nabla_x \eta)(x, l) \cdot \nabla p dx - \frac{\mu}{(1+t)^\lambda} \int_{|x|>l} (\nabla_x \eta)(x, l) \cdot (\rho u)(t, x) dx, \end{aligned} \quad (3.2)$$

where  $\nabla_x \eta(x, l) = |x|^{-3}(|x|^2 - l^2)x$  that vanishes on  $|x| = l$ . Integration by parts implies that

$$\begin{aligned} \partial_t^2 P(t, l) + \frac{\mu}{(1+t)^\lambda} \partial_t P(t, l) &= \sum_{i,j} \int_{|x|>l} (\partial_{x_i x_j} \eta) \rho u_i u_j dx + \int_{|x|>l} (\Delta \eta) p dx \\ &\equiv J_1(t, l) + \int_{|x|>l} (\Delta \eta) p dx. \end{aligned}$$

A direct computation of  $\partial_{x_i x_j} \eta$  shows that

$$\begin{aligned} J_1(t, l) &= \int_{|x|>l} \frac{2l^2}{|x|^3} \rho \left( \frac{x}{|x|} \cdot u \right)^2 dx \\ &\quad - \int_{|x|>l} \frac{|x|^2 - l^2}{|x|^3} \rho \left( \frac{x}{|x|} \cdot u \right)^2 dx + \int_{|x|>l} \frac{|x|^2 - l^2}{|x|^3} \rho |u|^2 dx \geq 0. \end{aligned} \quad (3.3)$$

Together with the fact that  $\Delta \eta = 2|x|^{-1} \geq 0$ , this yields

$$\partial_t^2 P(t, l) + \frac{\mu}{(1+t)^\lambda} \partial_t P(t, l) \geq 0.$$

Note that assumptions (1.8) and (1.9) imply  $P(0, l) > 0$  and  $\partial_t P(0, l) \geq 0$  for  $M_0 \leq l \leq M$ . Integrating the above differential inequality twice yields  $P(t, l) \geq 0$  for  $l \geq M_0$  and  $t \geq 0$ .

Next we derive a lower bound of  $P(t, l)$ . We now rewrite (3.2) as

$$\begin{aligned} \partial_t^2 P(t, l) = & - \sum_{i,j} \int_{|x|>l} (\partial_{x_i} \eta) \partial_{x_j} (\rho u_i u_j) dx - \int_{|x|>l} (\nabla_x \eta)(x, l) \cdot \nabla (p - \bar{p}) dx \\ & - \frac{\mu}{(1+t)^\lambda} \int_{|x|>l} (\nabla_x \eta)(x, l) \cdot (\rho u)(t, x) dx, \end{aligned}$$

where  $\bar{p} = p(\bar{\rho})$ . Let  $J_2(t, l) \equiv \int_{|x|>l} (\Delta \eta)(p - \bar{p}) dx$ . Then the above equation reads

$$\partial_t^2 P(t, l) + \frac{\mu}{(1+t)^\lambda} \partial_t P(t, l) = J_1(t, l) + J_2(t, l). \quad (3.4)$$

Note that

$$\partial_l^2 \eta(x, l) = 2|x|^{-1} = \Delta_x \eta(x, l).$$

Then

$$J_2(t, l) = \int_{|x|>l} \partial_l^2 \eta(x, l) (p(t, x) - \bar{p}) dx = \partial_l^2 \int_{|x|>l} \eta(x, l) (p(t, x) - \bar{p}) dx, \quad (3.5)$$

where we have used the fact that  $\eta$  and  $\partial_l \eta$  vanish on  $|x| = l$ . Combining (3.3)-(3.5), we arrive at

$$\partial_t^2 P(t, l) - \partial_l^2 P(t, l) + \frac{\mu}{(1+t)^\lambda} \partial_t P(t, l) \geq G(t, l), \quad (3.6)$$

where

$$G(t, l) = \partial_l^2 \int_{|x|>l} \eta(x, l) (p - \bar{p} - (\rho - \bar{\rho})) dx = \int_{|x|>l} 2|x|^{-1} (p - \bar{p} - (\rho - \bar{\rho})) dx. \quad (3.7)$$

Thanks to  $\gamma = 2$  and the sound speed  $\bar{c} = \sqrt{2A\bar{\rho}} = 1$ , we have

$$p - \bar{p} - (\rho - \bar{\rho}) = A(\rho^2 - \bar{\rho}^2 - 2\bar{\rho}(\rho - \bar{\rho})) = A(\rho - \bar{\rho})^2. \quad (3.8)$$

Substituting (3.8) into (3.7) gives

$$G(t, l) \geq 0.$$

To estimate  $P = P(t, l)$  from inequality (3.6), we first study the solution of the equation

$$\partial_t^2 \tilde{P}(t, l) - \partial_l^2 \tilde{P}(t, l) + \frac{\mu}{(1+t)^\lambda} \partial_t \tilde{P}(t, l) = G(t, l),$$

where  $\tilde{P}(0, l) = P(0, l)$  and  $\partial_t \tilde{P}(0, l) = \partial_t P(0, l)$ .

Rewriting the above equation as

$$\partial_t^2 \tilde{P}(t, l) - \partial_l^2 \tilde{P}(t, l) + \frac{\mu}{(1+t)^\lambda} (\partial_t \tilde{P}(t, l) - \partial_l \tilde{P}(t, l)) = G(t, l) - \frac{\mu}{(1+t)^\lambda} \partial_l \tilde{P}(t, l),$$

by the method of characteristics we have, for  $l \geq t \geq 0$ ,

$$\begin{aligned} \tilde{P}(t, l) = & \frac{1}{2} P(0, l+t) + \frac{1}{2\beta(t)} P(0, l-t) + \frac{1}{2} \int_0^t \frac{1}{\beta(\tau)} \frac{\mu}{(1+\tau)^\lambda} P(0, l+t-2\tau) d\tau \\ & + \int_0^t \frac{1}{\beta(\tau)} \partial_t P(0, l+t-2\tau) d\tau + \frac{1}{2} \int_0^t \int_{l-t+\tau}^{l+t-\tau} \frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)} G(\tau, y) dy d\tau \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \int_0^t \frac{\beta(\tau)}{\beta(t)} \frac{\mu}{(1+\tau)^\lambda} \tilde{P}(\tau, l-t+\tau) d\tau \\
& + \frac{1}{2} \int_0^t \int_\tau^t \frac{\beta(\tau)}{\beta(s)} \frac{\mu^2}{(1+\tau)^\lambda (1+s)^\lambda} \tilde{P}(\tau, l+t-2s+\tau) ds d\tau \\
& - \frac{1}{2} \int_0^t \frac{\mu}{(1+\tau)^\lambda} \tilde{P}(\tau, l+t-\tau) d\tau,
\end{aligned}$$

see (1.10). Together with assumptions (1.8)-(1.9) this yields, for  $l \geq t + M_0$ ,

$$\begin{aligned}
P(t, l) & \geq \tilde{P}(t, l) \geq \frac{1}{2\beta(t)} q_0(l-t) \\
& + \frac{1}{2} \int_0^t \int_{l-t+\tau}^{l+t-\tau} \frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)} G(\tau, y) dy d\tau - \frac{1}{2} \int_0^t \frac{\mu}{(1+\tau)^\lambda} P(\tau, l+t-\tau) d\tau. \quad (3.9)
\end{aligned}$$

Define the function

$$F(t) \equiv \int_0^t (t-\tau) \int_{\tau+M_0}^{\tau+M} P(\tau, l) \frac{dl}{l} d\tau. \quad (3.10)$$

Then, by (3.9), we have that

$$\begin{aligned}
F''(t) & = \int_{t+M_0}^{t+M} P(t, l) \frac{dl}{l} \geq \frac{1}{2\beta(t)} \int_{t+M_0}^{t+M} q_0(l-t) \frac{dl}{l} \\
& + \frac{1}{2} \int_{t+M_0}^{t+M} \int_0^t \int_{l-t+\tau}^{l+t-\tau} \frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)} G(\tau, y) dy d\tau \frac{dl}{l} \\
& - \frac{1}{2} \int_{t+M_0}^{t+M} \int_0^t \frac{\mu}{(1+\tau)^\lambda} P(\tau, l+t-\tau) d\tau \frac{dl}{l} \equiv J_3 + J_4 - J_5. \quad (3.11)
\end{aligned}$$

From  $\lambda > 1$  and assumption (1.8), we see that

$$J_3 \geq \frac{c_1}{t+M} \int_{t+M_0}^{t+M} q_0(l-t) dl = \frac{c_1}{t+M} \int_{M_0}^M q_0(l) dl = \frac{c_2 \varepsilon}{t+M} \quad (3.12)$$

where  $c_1, c_2 > 0$  are constants independent of  $\varepsilon$ . Note that  $P(\tau, y)$  is supported in  $\{y: y \leq \tau + M\}$  and nonnegative for  $y \geq M_0$ . Hence, there exists a constant  $C_1 > 0$  such that

$$J_5 \leq \frac{C_1}{(1+t)^\lambda} \int_0^t \int_{\tau+M_0}^{\tau+M} P(\tau, y) \frac{dy}{y} d\tau = \frac{C_1}{(1+t)^\lambda} F'(t). \quad (3.13)$$

Substituting (3.13) into (3.11) gives

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq J_3 + J_4. \quad (3.14)$$

To bound  $J_4$  from below, we write

$$J_4 = \frac{1}{2} \int_0^{t-M_1} \int_{\tau+M_0}^{\tau+M} G(\tau, y) \int_{t+M_0}^{y+t-\tau} \frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)} \frac{dl}{l} dy d\tau$$

$$\begin{aligned}
& + \frac{1}{2} \int_{t-M_1}^t \int_{\tau+M_0}^{2t-\tau+M_0} G(\tau, y) \int_{t+M_0}^{y+t-\tau} \frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)} \frac{dl}{l} dy d\tau \\
& + \frac{1}{2} \int_{t-M_1}^t \int_{2t-\tau+M_0}^{\tau+M} G(\tau, y) \int_{y-t+\tau}^{y+t-\tau} \frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)} \frac{dl}{l} dy d\tau \\
& \equiv J_{4,1} + J_{4,2} + J_{4,3},
\end{aligned} \tag{3.15}$$

where  $M_1 = (M - M_0)/2$ . For  $t < M_1$ ,  $t - M_1$  in the limits of integration is replaced by 0. By  $\lambda > 1$ , for the integrand in  $J_{4,1}$  we have that

$$\int_{t+M_0}^{y+t-\tau} \frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)} \frac{dl}{l} \geq c \frac{y - \tau - M_0}{t + M} \geq c \frac{(t - \tau)(y - \tau - M_0)^2}{(t + M)^2}. \tag{3.16}$$

Analogously, for the integrands in  $J_{4,2}$  and  $J_{4,3}$  we have that

$$\int_{t+M_0}^{y+t-\tau} \frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)} \frac{dl}{l} \geq c \frac{(t - \tau)(y - \tau - M_0)^2}{(t + M)^2} \tag{3.17}$$

and

$$\int_{y-t+\tau}^{y+t-\tau} \frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)} \frac{dl}{l} \geq c \frac{t - \tau}{t + M} \geq c \frac{(t - \tau)(y - \tau - M_0)^2}{(t + M)^2}, \tag{3.18}$$

where  $c > 0$  is a constant. Substituting (3.16)-(3.18) into (3.15) yields

$$J_4 \geq \frac{c}{(t + M)^2} \int_0^t (t - \tau) \int_{\tau+M_0}^{\tau+M} (y - \tau - M_0)^2 \partial_y^2 \tilde{G}(\tau, y) dy d\tau,$$

where  $\tilde{G}(t, l) = \int_{|x|>l} \eta(x, l) (p - \bar{p} - (\rho - \bar{\rho})) dx$ . Note that  $\tilde{G}(\tau, y) = \partial_y \tilde{G}(\tau, y) = 0$  for  $y = \tau + M$ . Thus, it follows from the integration by parts together with (3.7)-(3.8) that

$$\begin{aligned}
J_4 & \geq \frac{c}{(t + M)^2} \int_0^t (t - \tau) \int_{\tau+M_0}^{\tau+M} \tilde{G}(\tau, y) dy d\tau \\
& \geq \frac{c}{(t + M)^2} \int_0^t (t - \tau) \int_{\tau+M_0}^{\tau+M} \int_{|x|>y} \eta(x, y) (\rho(\tau, x) - \bar{\rho})^2 dx dy d\tau \\
& \equiv \frac{c}{(t + M)^2} J_6.
\end{aligned} \tag{3.19}$$

By applying the Cauchy-Schwartz inequality to  $F(t)$  defined by (3.10), we arrive at

$$F^2(t) \leq J_6 \int_0^t (t - \tau) \int_{\tau+M_0}^{\tau+M} \int_{y<|x|<\tau+M} \eta(x, y) dx \frac{dy}{y^2} d\tau \equiv J_6 J_7. \tag{3.20}$$

We estimate  $J_7$  as

$$J_7 = \int_0^t (t - \tau) \int_{\tau+M_0}^{\tau+M} \int_{y<|x|<\tau+M} \frac{(|x| - y)^2}{|x|} dx \frac{dy}{y^2} d\tau$$

$$\begin{aligned}
&= \int_0^t (t - \tau) \int_{\tau+M_0}^{\tau+M} \int_y^{\tau+M} 4\pi l (l - y)^2 dl \frac{dy}{y^2} d\tau \\
&\leq C \int_0^t (t - \tau) \int_{\tau+M_0}^{\tau+M} (\tau + M) (\tau + M - y)^3 \frac{dy}{y^2} d\tau \\
&\leq C \int_0^t (t - \tau) (\tau + M) \int_{\tau+M_0}^{\tau+M} \frac{dy}{y^2} d\tau \\
&\leq C \int_0^t \frac{t - \tau}{\tau + M} d\tau \leq C (t + M) \log \frac{t}{M + 1}.
\end{aligned} \tag{3.21}$$

Combining (3.12), (3.14) and (3.19)-(3.21) gives the ordinary differential inequalities

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq \frac{c_2 \varepsilon}{t + M}, \quad t \geq 0, \tag{3.22}$$

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq C (t + M)^3 \log \frac{t}{M + 1} F^2(t), \quad t \geq 0. \tag{3.23}$$

Next, we apply (3.22)-(3.23) to prove that the lifespan  $T_\varepsilon$  of smooth solution  $F(t)$  is finite for all  $0 < \varepsilon \leq \varepsilon_0$ . The fact that  $F(0) = F'(0) = 0$ , together with (3.22), yields

$$F'(t) \geq C\varepsilon \log(t/M + 1), \quad t \geq 0, \tag{3.24}$$

$$F(t) \geq C\varepsilon(t + M) \log(t/M + 1), \quad t \geq t_1 \equiv Me^2, \tag{3.25}$$

where the constant  $C > 0$  is independent of  $\varepsilon$ . Substituting (3.25) into (3.23) derives

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq C\varepsilon^2(t + M)^{-1} \log(t/M + 1), \quad t \geq t_1,$$

which leads to the improvement

$$F(t) \geq C\varepsilon^2(t + M) \log^2(t/M + 1), \quad t \geq t_2 \equiv Me^3 > t_1. \tag{3.26}$$

Substituting this into (3.23) derives

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq C\varepsilon^2(t + M)^{-2} \log(t/M + 1) F(t), \quad t \geq t_2. \tag{3.27}$$

It follows from (3.24) that  $F'(t) \geq 0$  for  $t \geq 0$ . Then multiplying (3.27) by  $F'(t)$  and integrating from  $t_3$  (which will be chosen later) to  $t$  yield

$$F'(t)^2 \geq C_2 F'(t_3)^2 + C_3 \varepsilon^2 \int_{t_3}^t (s + M)^{-2} \log(s/M + 1) [F(s)^2]' ds.$$

Integrating by parts yields

$$\begin{aligned}
F'(t)^2 &\geq C_2 F'(t_3)^2 + C_3 \varepsilon^2 (t + M)^{-2} \log(t/M + 1) F(t)^2 - (t_3 + M)^{-2} \log(t_3/M + 1) F(t_3)^2 \\
&\quad - \int_{t_3}^t \left( \frac{\log(s/M + 1)}{(s + M)^2} \right)' F(s)^2 ds, \quad t \geq t_3,
\end{aligned} \tag{3.28}$$

where  $\left(\frac{\log(s/M+1)}{(s+M)^2}\right)' \leq 0$  for  $t \geq t_3 \geq t_2$ . On the other hand, (3.22) implies

$$\left(e^{-\frac{C_1}{\lambda-1}[(1+t)^{1-\lambda}-1]}F'(t)\right)' \geq 0, \quad t \geq 0,$$

which yields for  $0 \leq t \leq \tau$

$$F'(t) \leq e^{\frac{C_1}{\lambda-1}[(1+t)^{1-\lambda}-(1+\tau)^{1-\lambda}]}F'(\tau). \quad (3.29)$$

Together with  $F(0) = 0$ , this yields

$$F(t) = \int_0^t F'(s)ds \leq C_4 t F'(t), \quad t > 0. \quad (3.30)$$

Choose

$$t_3 = M \left( e^{\frac{C_2}{2C_3C_4\varepsilon^2}} - 1 \right) \quad (3.31)$$

which satisfies  $2C_3C_4 \log(t_3/M + 1)\varepsilon^2 = C_2$ . Together with (3.28) and (3.30), this yields

$$F'(t) \geq \sqrt{C_3}\varepsilon(t+M)^{-1} \log^{\frac{1}{2}}(t/M+1)F(t), \quad t \geq t_3. \quad (3.32)$$

By integrating (3.32) from  $t_3$  to  $t$ , we arrive at

$$\log \frac{F(t)}{F(t_3)} \geq C\varepsilon \log^{\frac{3}{2}}\left(\frac{t+M}{t_3+M}\right), \quad t \geq t_3.$$

If  $t \geq t_4 \equiv Ct_3^2$ , then we have

$$\log \frac{F(t)}{F(t_3)} \geq 8 \log(t/M+1).$$

Together with (3.26) for  $F(t_3)$ , this yields

$$F(t) \geq C\varepsilon^2(t+M)^8, \quad t \geq t_4. \quad (3.33)$$

Substituting this into (3.23) derives

$$F''(t) + \frac{C_1}{(1+t)^\lambda}F'(t) \geq C\varepsilon F(t)^{\frac{3}{2}}, \quad t \geq t_4.$$

Multiplying this differential inequality by  $F'(t)$  and integrating from  $t_4$  to  $t$  yields

$$F'(t)^2 \geq C\varepsilon \left( F(t)^{\frac{5}{2}} - F(t_4)^{\frac{5}{2}} \right).$$

On the other hand, (3.29) and (3.30) imply that, for  $t \geq t_4$ ,

$$F(t) = F'(\xi)(t-t_4) + F(t_4) \geq CF'(t_4)(t-t_4) \geq CF(t_4)\frac{t-t_4}{t_4},$$

where  $t_4 \leq \xi \leq t$ . If  $t \geq t_5 \equiv Ct_4$ , then we have

$$F(t)^{\frac{5}{2}} - F(t_4)^{\frac{5}{2}} \geq \frac{1}{2}F(t)^{\frac{5}{2}}.$$

Thus

$$F'(t) \geq C\sqrt{\varepsilon}F(t)^{\frac{5}{4}}, \quad t \geq t_5. \quad (3.34)$$

If  $T_\varepsilon > 2t_5$ , then integrating (3.34) from  $t_5$  to  $T_\varepsilon$  derives

$$F(t_5)^{-\frac{1}{4}} - F(T_\varepsilon)^{-\frac{1}{4}} \geq C\sqrt{\varepsilon}T_\varepsilon.$$

We see from (3.33) and  $t_5 = Ct_3^2$  that

$$F(t_5) \geq C\varepsilon^2 e^{\frac{C}{\varepsilon^2}},$$

which together with  $F(T_\varepsilon) > 0$  is a contradiction. Thus,  $T_\varepsilon \leq 2t_5 = Ct_3^2$ . From the choice of  $t_3$  in (3.31), we see that  $T_\varepsilon \leq e^{C/\varepsilon^2}$ .

### Case 2: $\gamma > 1$ and $\gamma \neq 2$ .

Recall that the sound speed is  $\bar{c} = \sqrt{\gamma A \bar{\rho}^{\gamma-1}} = 1$ . Instead of (3.8) we have

$$p - \bar{p} - (\rho - \bar{\rho}) = A(\rho^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1}(\rho - \bar{\rho})) \equiv A\psi(\rho, \bar{\rho}).$$

The convexity of  $\rho^\gamma$  for  $\gamma > 1$  implies that  $\psi(\rho, \bar{\rho})$  is positive for  $\rho \neq \bar{\rho}$ . Applying Taylor's theorem, we have

$$\psi(\rho, \bar{\rho}) \geq C(\gamma, \bar{\rho}) \Phi_\gamma(\rho, \bar{\rho}),$$

where  $C(\gamma, \bar{\rho})$  is a positive constant and  $\Phi_\gamma$  is given by

$$\Phi_\gamma(\rho, \bar{\rho}) = \begin{cases} (\bar{\rho} - \rho)^\gamma, & \rho < \frac{1}{2}\bar{\rho}, \\ (\rho - \bar{\rho})^2, & \frac{1}{2}\bar{\rho} \leq \rho \leq 2\bar{\rho}, \\ (\rho - \bar{\rho})^\gamma, & \rho > 2\bar{\rho}. \end{cases}$$

For  $\gamma > 2$ , we have that  $(\bar{\rho} - \rho)^\gamma = (\bar{\rho} - \rho)^2(\bar{\rho} - \rho)^{\gamma-2} \geq C(\gamma, \bar{\rho})(\rho - \bar{\rho})^2$  for  $2\rho < \bar{\rho}$  and  $(\rho - \bar{\rho})^\gamma = (\rho - \bar{\rho})^2(\rho - \bar{\rho})^{\gamma-2} \geq C(\gamma, \bar{\rho})(\rho - \bar{\rho})^2$  for  $\rho > 2\bar{\rho}$ . Thus,  $\psi(\rho, \bar{\rho}) \geq C(\gamma, \bar{\rho})(\rho - \bar{\rho})^2$ . In this case, Theorem 1.2 can be shown completely analogously to Case 1.

Next we treat the case  $1 < \gamma < 2$ . We define  $F(t)$  as in (3.10),

$$F(t) = \int_0^t \int_{\tau+M_0}^{\tau+M} \frac{1}{l} \int_{|x|>l} \frac{(|x|-l)^2}{|x|} (\rho(\tau, x) - \bar{\rho}) dx dl d\tau.$$

Similarly to the case of  $\gamma = 2$ , we have

$$F''(t) \geq J_3 + J_4 - J_5, \quad (3.35)$$

where

$$\begin{aligned} J_3 &\geq \frac{C\varepsilon}{t+M}, \\ J_4 &\geq C(t+M)^{-2} \tilde{J}_6, \\ J_5 &\leq \frac{C_1}{(1+t)^\lambda} F'(t), \end{aligned}$$

and

$$\tilde{J}_6 = \int_0^t (t - \tau) \int_{\tau+M_0}^{\tau+M} \int_{|x|>y} \frac{(|x| - y)^2}{|x|} \Phi_\gamma(\rho(\tau, x) - \bar{\rho}) dx dy d\tau.$$

Denote  $\Omega_1 = \{(\tau, x) : \bar{\rho} \leq \rho(\tau, x) \leq 2\bar{\rho}\}$ ,  $\Omega_2 = \{(\tau, x) : \rho(\tau, x) > 2\bar{\rho}\}$ , and  $\Omega_3 = \{(\tau, x) : \rho(\tau, x) < \bar{\rho}\}$ . Divide  $F(t)$  into a sum the three integrals over the domains  $\Omega_i$  ( $1 \leq i \leq 3$ )

$$F(t) = F_1(t) + F_2(t) + F_3(t) \equiv \int_{\Omega_1} \cdots + \int_{\Omega_2} \cdots + \int_{\Omega_3} \cdots$$

Corresponding to the three parts of  $F(t)$ , we define  $\tilde{J}_6 \equiv \tilde{J}_{6,1} + \tilde{J}_{6,2} + \tilde{J}_{6,3}$ . In view of  $F(t) \geq 0$  and  $F_3(t) \leq 0$ , we have

$$F(t) \leq F_1(t) + F_2(t).$$

Applying Hölder's inequality for the domains  $\Omega_1$  and  $\Omega_2$ , we obtain that

$$\begin{aligned} F(t) &\leq \tilde{J}_{6,1}^{\frac{1}{2}} \left( \int_0^t (t - \tau) \int_{\tau+M_0}^{\tau+M} \frac{1}{y^2} \int_{y<|x|\leq\tau+M} \frac{(|x| - y)^2}{|x|} dx dy d\tau \right)^{\frac{1}{2}} \\ &\quad + \tilde{J}_{6,2}^{\frac{1}{\gamma}} \left( \int_0^t (t - \tau) \int_{\tau+M_0}^{\tau+M} \frac{1}{y^{\frac{\gamma}{\gamma-1}}} \int_{y<|x|\leq\tau+M} \frac{(|x| - y)^2}{|x|} dx dy d\tau \right)^{\frac{\gamma-1}{\gamma}} \\ &\leq \tilde{J}_6^{\frac{1}{2}} (t + M)^{\frac{1}{2}} \log^{\frac{1}{2}}(t/M + 1) + \tilde{J}_6^{\frac{1}{\gamma}} (t + M)^{\frac{\gamma-1}{\gamma}} \\ &= (\tilde{J}_6(t + M)^{-1})^{\frac{1}{2}} (t + M) \log^{\frac{1}{2}}(t/M + 1) + (\tilde{J}_6(t + M)^{-1})^{\frac{1}{\gamma}} (t + M). \end{aligned}$$

In view of  $1 < \gamma < 2$ , we have  $\frac{1}{2\gamma} < \frac{1}{2} < \frac{1}{\gamma}$ . Applying Young's inequality yields

$$F(t) \leq \left( (\tilde{J}_6(t + M)^{-1})^{\frac{1}{2\gamma}} + (\tilde{J}_6(t + M)^{-1})^{\frac{1}{\gamma}} \right) (t + M) \log^{\frac{1}{2}}(t/M + 1), \quad t \geq \tilde{t}_1 \equiv Me.$$

Together with the fact that  $F(t) \geq C\varepsilon(t + M) \log(t/M + 1)$ , this yields

$$\tilde{J}_6 \geq CF(t)^\gamma (t + M)^{1-\gamma} \log^{-\frac{\gamma}{2}}(t/M + 1), \quad t \geq \tilde{t}_1.$$

Substituting this into (3.35) yields

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq \frac{C\varepsilon}{t+M}, \quad t \geq 0, \quad (3.36)$$

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq CF(t)^\gamma (t + M)^{-1-\gamma} \log^{-\frac{\gamma}{2}}(t/M + 1), \quad t \geq \tilde{t}_1. \quad (3.37)$$

Substituting  $F(t) \geq C\varepsilon(t + M) \log(t/M + 1)$  into (3.37) yields

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq C\varepsilon^\gamma (t + M)^{-1} \log^{\frac{\gamma}{2}}(t/M + 1).$$

Integrating this yields

$$F(t) \geq C\varepsilon^\gamma (t + M) \log^{\frac{\gamma+2}{2}}(t/M + 1).$$



Substituting this into (3.37) again gives

$$\begin{aligned} F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \\ \geq C\varepsilon^{\gamma^2} (t+M)^{-1} \log^{\frac{\gamma(\gamma+1)}{2}}(t/M+1) = C\varepsilon^{\gamma^2} (t+M)^{-1} \log^{\frac{\gamma(\gamma^2-1)}{2(\gamma-1)}}(t/M+1). \end{aligned}$$

Repeating this process  $n$  times, we see that

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq C\varepsilon^{\gamma^n} (t+M)^{-1} \log^{\frac{\gamma(\gamma^n-1)}{2(\gamma-1)}}(t/M+1), \quad (3.38)$$

where  $n = \lceil \log_\gamma 2 \rceil$ . Solving (3.38) yields

$$F(t) \geq C\varepsilon^{\gamma^n} (t+M) \log^{\frac{\gamma(\gamma^n-1)}{2(\gamma-1)}+1}(t/M+1), \quad t \geq \tilde{t}_2,$$

where  $\tilde{t}_2 > 0$  is a constant only depending on  $\gamma$ . Substituting this into (3.37) derives

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq CF(t)\varepsilon^{\gamma^n(\gamma-1)} (t+M)^{-1} \log^{\frac{\gamma^{n+1}-2}{2}}(t/M+1), \quad t \geq \tilde{t}_2, \quad (3.39)$$

where  $\frac{\gamma^{n+1}-2}{2} > 0$  by the choice of  $n = \lceil \log_\gamma 2 \rceil$ . Since (3.39) is analogous to (3.27), as in Case 1, we can choose  $\tilde{t}_3 = O\left(e^{C\varepsilon^{-\frac{2\gamma^n(\gamma-1)}{\gamma^{n+1}-2}}}\right)$  such that

$$F'(t) \geq C\varepsilon^{\frac{\gamma^n(\gamma-1)}{2}} (t+M)^{-1} \log^{\frac{\gamma^{n+1}-2}{4}}(t/M+1) F(t), \quad t \geq \tilde{t}_3,$$

which is similar to (3.32) and yields

$$F(t) \geq C\varepsilon^{C_\gamma} (t+M)^{\frac{2(\gamma+2)}{\gamma-1}}, \quad t \geq \tilde{t}_4 \equiv C\tilde{t}_3, \quad (3.40)$$

where  $C_\gamma > 0$  is a constant depending on  $\gamma$ . Substituting (3.40) into (3.37) yields

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq C\varepsilon^{C_\gamma} F(t)^{\frac{\gamma+1}{2}}, \quad t \geq \tilde{t}_4. \quad (3.41)$$

Multiplying (3.41) by  $F'(t)$  and integrating over the variable  $t$  as in Case 1, we have

$$F'(t) \geq C\varepsilon^{C_\gamma} F(t)^{\frac{\gamma+3}{4}}, \quad t \geq \tilde{t}_5 \equiv C\tilde{t}_4.$$

Together with  $\gamma > 1$  and the choice of  $\tilde{t}_3$ , this yields  $T_\varepsilon < \infty$ .

Both Case 1 and Case 2 complete the proof of Theorem 1.2.  $\square$

## 4 Blowup for large data

In this section, we establish a blowup result for large amplitude smooth solutions of Eq. (1.1) which is valid for all  $\lambda \geq 0$ . More precisely, instead of (1.1) we consider the Cauchy problem

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + pI_3) = -\frac{\mu}{(1+t)^\lambda} \rho u, \\ \rho(0, x) = \bar{\rho} + \tilde{\rho}_0(x), \quad u(0, x) = \tilde{u}_0(x), \end{cases} \quad (4.1)$$

where  $\tilde{\rho}_0, \tilde{u}_0 \in C_0^\infty(\mathbb{R}^3)$ ,  $\text{supp } \tilde{\rho}_0, \text{supp } \tilde{u}_0 \subseteq B(0, M) \equiv \{x: |x| \leq M\}$ , and  $\rho(0, \cdot) > 0$ . Motivated by the treatment of the special case of  $\lambda = 0$  in [19], we introduce the functions

$$\begin{aligned} H(t) &\equiv \int_{\mathbb{R}^3} x \cdot (\rho u)(t, x) dx, & L(t) &\equiv \int_{\mathbb{R}^3} (\rho(t, x) - \bar{\rho}) dx, \\ \alpha(t) &\equiv (t + M)^2 \left( L(0) + \frac{4\pi^2 \bar{\rho}}{3} (t + M)^3 \right), \end{aligned}$$

and also remind the reader of the definition of the function  $\beta$  in (1.10).

Then we have the following result:

**Theorem 4.1.** *Suppose that  $L(0) \geq 0$  and*

$$H(0) \int_0^{T^*} \frac{d\tau}{\alpha(\tau)\beta(\tau)} > 1. \quad (4.2)$$

*for some  $T^* > 0$ . Then  $T < T^*$  holds for any solution  $(\rho, u) \in C^1([0, T] \times \mathbb{R}^3)$  of (4.1).*

*Proof.* From the first equation of (4.1), we see that

$$L'(t) = - \int_{\mathbb{R}^3} \text{div}(\rho u) dx = 0,$$

which implies  $L(t) = L(0)$ . Applying the second equation of (4.1), we find that

$$H'(t) = \int_{\mathbb{R}^3} x \cdot \partial_t(\rho u)(t, x) dx = \int_{\mathbb{R}^3} x \cdot \left[ -\text{div}(\rho u \otimes u) - \nabla p - \frac{\mu}{(1+t)^\lambda} \rho u \right] dx.$$

An integration by parts gives

$$H'(t) + \frac{\mu}{(1+t)^\lambda} H(t) = \int_{\mathbb{R}^3} (\rho|u|^2 + 3(p(\rho) - p(\bar{\rho}))) dx. \quad (4.3)$$

Note that the convexity of  $p = A\rho^\gamma$  for  $\gamma > 1$  and  $c(\bar{\rho}) = 1$  imply that

$$\int_{\mathbb{R}^3} (p(\rho) - p(\bar{\rho})) dx \geq \int_{\mathbb{R}^3} A\gamma\bar{\rho}^{\gamma-1}(\rho - \bar{\rho}) dx = L(0). \quad (4.4)$$

Furthermore, by applying the Cauchy-Schwartz inequality to  $H(t)$  and taking into account  $\text{supp } u(t, \cdot) \subseteq B(0, M+t)$  for any fixed  $t \geq 0$ , we have

$$\begin{aligned} H(t)^2 &\leq \left( \int_{\mathbb{R}^3} \rho|u|^2 dx \right) \left( \int_{|x| \leq t+M} \rho|x|^2 dx \right) \\ &\leq (t+M)^2 \left( L(0) + \frac{4\pi^2 \bar{\rho}}{3} (t+M)^3 \right) \int_{\mathbb{R}^3} \rho|u|^2 dx = \alpha(t) \int_{\mathbb{R}^3} \rho|u|^2 dx. \end{aligned} \quad (4.5)$$

Substituting (4.4)-(4.5) into (4.3) yields

$$H'(t) + \frac{\mu}{(1+t)^\lambda} H(t) \geq \frac{H(t)^2}{\alpha(t)} + 3L(0).$$

Together with  $L(0) \geq 0$  and  $H(0) > 0$  due to (4.2), this shows that  $H(t) > 0$  for all  $t \in [0, T]$ . Denoting  $G(t) \equiv \beta(t)H(t)$ , from (1.10) and (4) we then get that

$$G'(t) \geq \frac{G^2(t)}{\alpha(t)\beta(t)}. \quad (4.6)$$

Now suppose that  $T \geq T^*$ . Then integrating (4.6) from 0 to  $T$  yields

$$\frac{1}{H(0)} - \frac{1}{G(T)} \geq \int_0^T \frac{d\tau}{\alpha(\tau)\beta(\tau)} \geq \int_0^{T^*} \frac{d\tau}{\alpha(\tau)\beta(\tau)}$$

which is a contradiction in view of  $G(T) > 0$  and (4.2).

Thus, Theorem 4.1 has been proved.  $\square$

*Acknowledgement.* Yin Huicheng wishes to express his gratitude to Professor Michael Reissig, Technical University Bergakademie Freiberg, Germany, for his interests in this problem and some fruitful discussions in the past.

## References

- [1] S. Alinhac, *Blowup of small data solutions for a quasilinear wave equation in two space dimensions*. Ann. of Math. (2) **149** (1999), 97–127.
- [2] S. Alinhac, *Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions. II*. Acta Math. **182** (1999), 1–23.
- [3] S. Alinhac, *Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux*. Invent. Math. **111** (1993), 627–670.
- [4] D. Christodoulou, *The formation of shocks in 3-dimensional fluids*. EMS Monogr. Math., Eur. Math. Soc., Zürich, 2007.
- [5] D. Christodoulou and Miao Shuang, *Compressible flow and Euler's equations*. Surv. Mod. Math., vol. 9, Int. Press, Somerville, MA, 2014.
- [6] D. Christodoulou and A. Lisibach, *Shock development in spherical symmetry*. arXiv:1501.04235 (2015).
- [7] Ding Bingbing, I. Witt, and Yin Huicheng, *On small data solutions of general 3-D quasilinear wave equations. II*. Preprint (2014).
- [8] M. D'Abbico and M. Reissig, *Semilinear structural damped waves*. Math. Methods Appl. Sci. **37** (2014), 1570–1592.
- [9] M. D'Abbico, S. Lucente, and M. Reissig, *Semi-linear wave equations with effective damping*. Chin. Ann. Math. Ser. B **34** (2013), 345–380.
- [10] W.N. do Nascimento and J. Wirth, *Wave equations with mass and dissipation*. Adv. Differential Equations **20** (2015), 661–696.

- [11] L. Hörmander, *Lectures on Nonlinear Hyperbolic Equations*. Math. Appl. (Berlin), vol. 26, Springer, Heidelberg, 1997.
- [12] S. Kawashima and Yong Wen-An, *Dissipative structure and entropy for hyperbolic systems of balance laws*. Arch. Rational Mech. Anal. **174** (2004), 345–364.
- [13] S. Klainerman, *Remarks on the global Sobolev inequalities in the Minkowski space  $\mathbb{R}^{n+1}$* . Comm. Pure Appl. Math. **40** (1987), 111–117.
- [14] S. Klainerman and T.C. Sideris, *On almost global existence for nonrelativistic wave equations in 3D*. Comm. Pure Appl. Math. **49** (1996), 307–321.
- [15] H. Lindblad, *On the lifespan of solutions of nonlinear wave equations with small initial data*. Comm. Pure Appl. Math. **43** (1990), 445–472.
- [16] Pan Ronghua and Zhao Kun, *The 3D compressible Euler equations with damping in a bounded domain*. J. Differential Equations **246** (2009), 581–596.
- [17] T. Sideris, *Delayed singularity formation in 2D compressible flow*. Amer. J. Math. **119** (1997), 371–422.
- [18] T. Sideris, *Formation of singularities in three-dimensional compressible fluids*. Comm. Math. Phys. **101** (1985), 475–485.
- [19] T. Sideris, B. Thomases, and Wang Dehua, *Long time behavior of solutions to the 3D compressible Euler equations with damping*. Comm. Partial Differential Equations **28** (2003), 795–816.
- [20] J. Speck, *Shock formation in small-data solutions to 3D quasilinear wave equations*. arXiv:1407.6320 (2014).
- [21] Tan Zhong and Wu Guochun, *Large time behavior of solutions for compressible Euler equations with damping in  $\mathbb{R}^3$* . J. Differential Equations **252** (2012), 1546–1561.
- [22] Wang Weike and Yang Tong, *The pointwise estimates of solutions for Euler equations with damping in multi-dimensions*. J. Differential Equations **173** (2001), 410–450.
- [23] J. Wirth, *Wave equations with time-dependent dissipation. I. Non-effective dissipation*. J. Differential Equations **222** (2006), 487–514.
- [24] J. Wirth, *Wave equations with time-dependent dissipation. II. Effective dissipation*. J. Differential Equations **232** (2007), 74–103.
- [25] Yin Huicheng and Qiu Qingjiu, *The blowup of solutions for 3-D axisymmetric compressible Euler equations*. Nagoya Math. J., **154** (1999), 157–169.
- [26] Yin Huicheng, *Formation and construction of a shock wave for 3-D compressible Euler equations with the spherical initial data*. Nagoya Math. J., **175** (2004), 125–164.